Bivariate degradation modeling and reliability analysis based on a shared frailty factor with truncated normal distribution

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ABSTRACT

For a complex product, a single performance characteristic(PC) often fails to fully reflect its degradation process, making it essential to consider the joint degradation of multiple PCs. In this paper, we propose a bivariate degradation model based on a shared frailty factor with the truncated normal distribution, using Wiener processes to characterize the marginal distributions of the PCs. The assumption of the truncated normal distribution aligns better with the physical background where the degradation rates of PCs are non-negative during actual degradation processes. Furthermore, a method for inferring unknown parameters is developed by employing the expectation maximization algorithm. Under this modeling assumption, it became possible to obtain an analytical expression for the product's lifetime distribution on the basis of the concept of the first hitting time. Therefore, in this paper, we further extend the normal distribution integral lemmas to the case of the truncated normal distribution, and provide analytical expression for the cumulative distribution function of the product lifetime. Finally, the rational effectiveness of the proposed model and methods is validated through a numerical simulation example and a case study on wheel wear.

1. INTRODUCTION

The demand for high-reliability products has become increasingly prominent in many fields. Performance degradation has become the main mode of failure for such products(Kawakubo, Miyazawa, Nagata, & Kobatake, 2003; Z. Wang, Hu, Wang, Zhou, & Si, 2014). Consequently, methods for modeling and reliability analysis based on degradation data of a single performance characteristic(PC) have been proposed and extensively studied(Lu & Meeker, 1993; Meeker & Escobar, 1998). Among these, the general path and the stochastic process are two main categories of performance degradation models. Given the inherent randomness in products, stochastic process-based modeling methods have garnered greater attention from scholars. The Wiener process, in particular, has been widely applied due to its clear physical significance and favorable mathematical properties(Whitmore, 1995; Z. Zhang, Si, Hu, & Lei, 2018).

As product functionality becomes more complex, its degradation usually can be characterized by two or more PCs(X. Wang, Balakrishnan, & Guo, 2015; Zhai & Ye, 2023). Additionally, these PCs often exhibit interdependencies. Consequently, the issue of multi-PC degradation has attracted considerable attention recently, with research on bivariate degradation problems being the most extensive(Peng, Li, Yang, Zhu, & Huang, 2016; Xu, Wang, Zhu, Pang, & Lian, 2024). Currently, the method based on the copula function is an important mean of addressing such issues(X. Zhang & Wilson, 2017; F. Wang & Li, 2017; Zheng, Chen, Lin, Ye, & Zhai, 2023). This method describes the marginal degradation processes and the relationship between PCs through single-PC degradation models and a copula function. Since these two aspects are considered separately, this approach does not impose restrictions on the types of marginal degradation processes, offering great flexibility in modeling. However, the selection of copula functions is based solely on certain information criteria in most cases, which may lack intuitive significance. And it is worth noticing that the lifetime distribution of the product in the sense of first hitting time(FHT) cannot be expressed analytically due to the complexity of copula functions.

To address the above issues, a new bivariate degradation model based on the shared frailty factor is proposed by Xu et al.(Xu, Shen, Wang, & Tang, 2018). The shared frailty factor was initially introduced in the field of biostatistics to capture the common risk faced by multiple populations(Hougaard, 2000; Michiels, Baujat, Mahé, Sargent, & Pignon, 2005). In the mathematical model, the shared frailty factor is typically assumed to be a random variable, representing the common random effect induced by factors that cannot be quantitatively described, which affect the different populations. Specifically, this random variable is often defined as following a distribution that is always positive with a mean of 1. This assumption is made because negative frailty factors have no practical meaning. And the assumption of a mean of 1 implies that the frailty factor has a "neutral" effect on the failure time, preventing the frailty factor from having either an excessively large or small influence on the model. Based on this concept, Xu et al.(Xu et al., 2018) introduce a shared frailty factor in the drift term of the Wiener processes to model the correlation between two PCs. More importantly, an analytical expression for the FHT's lifetime distribution can be obtained in reliability analysis under this model assumption. Subsequently, this method has attracted growing interest(Yan, Wang, & Ma, 2023; Song & Cui, 2022; Barui, Mitra, & Balakrishnan, 2024).

However, it is important to note that when using the Wiener process to establish the marginal degradation processes of the PCs, the shared frailty factor is typically assumed to follow a normal distribution(Xu et al., 2018; Yan et al., 2023). This assumption often does not reflect real-world scenarios and does not align with the physical significance of the shared frailty factor. On one hand, the degradation rates of products are generally typically either always positive or always negative, and a reversal of the degradation rates does not occur, such as crack propagation(Meeker & Escobar, 1998) and battery capacity decline(Zhu et al., 2022). On the other hand, the frailty factor, from a physical perspective, is expected to add a frail component to the degradation process, leading to accelerated or slowed product failure. Therefore, using a normal distribution to model the shared frailty factor often overlooks its non-negativity constraint. When the dispersion of the shared frailty factor is small, this assumption does not cause significant errors. However, in real-life scenarios, the shared random effect usually cannot be ignored. Using the normal distribution to model the shared frailty factor in such cases can lead to an inadequate representation of the product's degradation process and adversely affecting the accuracy of reliability analysis.

To tackle this problem, we propose an improved bivariate degradation model along with the corresponding parameter estimation method and further derive the analytical expression for the FHT's lifetime distribution. The main contributions of this paper are as follows: first, we consider the non-negativity constraint of the shared frailty factor by modeling it with the truncated normal distribution, combined with the linear Wiener process to establish a bivariate degradation model. Further, considering the unobservability of the shared frailty factor, we employ the expectation maximization(EM) algorithm to obtain the maximum likelihood estimates of the parameters. Based on this, we further extend the existing normal distribution integral theory to derive the analytical expression for the cumulative distribution function(CDF) of the FHT's lifetime distribution, from which the reliability function can also be easily obtained.

The remainder of this paper is organized as follows. In Section 2, we conduct a bivariate degradation model that utilizes the truncated normal distribution for the shared frailty factor and linear Wiener processes, along with the method for estimating unknown parameters using the EM algorithm. In Section 3, we present the derivation of the analytical expression for the product's lifetime distribution in the context of FHT. A numerical simulation example and a case study are provided to validate the proposed method in Section 4. Finally, Section 5 concludes the paper.

2. MODEL CONSTRUCTION

2.1. Model Description

Assuming that two PCs degrade simultaneously during the product's usage, and the sth PC is defined as $X_s(t)$, $s = 1, 2$. Generally, the linear degradation pattern is commonly observed in engineering products, and many nonlinear patterns can be transformed into the linear one through certain conversions. Further considering the inherent random effects of the product, we employ linear Wiener processes to describe the marginal degradation processes of the two PCs. For the two PCs of the same product, they are often influenced by the same external environment, internal coupling failure mechanisms, and similar manufacturing defects, which leads to the degradation of the two PCs not being completely independent. To describe the shared failure risk caused by unknown factors affecting the two PCs, a shared frailty factor is introduced. Since these risks are random, the frailty factor is modeled as a random variable. To avoid altering the inherent degradation patterns of the two PCs, its mean is set to 1. Therefore, the degradation model for the sth PC is as follows

$$\mathrm{M}_0 : X_s(t) = \alpha\mu_s t + \sigma_s B(t), s = 1, 2 \tag{1}$$

where μ_s and σ_s are the drift coefficient and the diffusion coefficient of the sth PC; $B(\cdot)$ is the standard Brownian motion; α is the shared frailty factor, representing the common random effect on the degradation rates of the two PCs caused by experiencing the same environment, load, or operating conditions. Therefore, α is typically modeled as a random variable. Consequently, the correlation between the two PCs is naturally described. At the same time, for each individual PC, the individual variability is also characterized.

In existing research, the shared frailty factor is typically assumed to follow a normal distribution. However, as mentioned earlier, the degradation rate of products is generally not negative in practical scenarios. Thus, it is usually assumed to be a positive random variable with a mean of 1. Under the normal distribution assumption, the probability of the shared frailty factor being negative is often considered negligible, which clearly contradicts objective reality and our expectation. This assumption leads to the model that inadequately describe the degradation behavior of the two PCs, resulting in irrationality in reliability analysis.

To ensure the non-negative nature of the shared frailty factor, we assume that it follows a truncated normal distribution, denoted as $\alpha \sim TN(\mu_\alpha, \sigma_\alpha^2)$. Its probability density function can be expressed as

$$f(\alpha) = \frac{\phi\left(\frac{\alpha-\mu_\alpha}{\sigma_\alpha}\right)}{\sigma_\alpha \Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)}, \quad \alpha > 0 \tag{2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the PDF and the CDF of the normal distribution, respectively. Correspondingly, the mean and the variance of α can be written as

$$\begin{aligned} E(\alpha) &= \mu_\alpha + \frac{\sigma_\alpha \phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)}{\Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)} \\ \mathrm{Var}(\alpha) &= \sigma_\alpha^2 - \sigma_\alpha^2 \frac{\phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)}{\Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)} \left(\frac{\phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)}{\Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)} + \frac{\mu_\alpha}{\sigma_\alpha} \right) \end{aligned} \tag{3}$$

It is important to note that, as previously stated, the mean of α is defined to be 1. Therefore, μ_α and σ_α can be solved correspondingly according to (3), indicating that there is only one unknown parameter in the distribution of α. Therefore, the unknown parameters to be estimated is $\Theta = \{\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_\alpha\}$. In the next subsection, we will provide the parameter estimation method.

2.2. Parameter Estimation

Effective model parameter estimation is very important for a reasonable reliability inference. For our proposed model, the complete likelihood function requires integration over α. The assumption that α follows the truncated normal distribution makes the expression of the complete likelihood function quite complex, making it difficult to directly maximize. To address this issue, we employ the EM algorithm, which has demonstrated excellent performance in the parameter estimation problem involving latent variables. The algorithm consists of the expectation-step(E-step) and the maximization-step(M-step). The E-step aims to obtain the expected log-likelihood function, thus avoiding the challenge mentioned above. And the M-step maximizes this expected log-likelihood function to estimate the model parameters. Iteratively applying the E-step and M-step continues until convergence, resulting in the final estimates of the unknown model parameters. During the iteration process, the objective function value of the EM algorithm is monotonic and bounded above, which contributes to the good convergence properties of the algorithm.

Assuming that a total of m units are involved in the test, each unit is jointly characterized by two PCs to represent its degradation. Therefore, the observed degradation can be denoted as x_{isj} for the sth PC of the ith unit at measurement time t_{isj}, $i = 1, 2, \ldots, m$, $s = 1, 2$, $j = 1, 2, \ldots, n_{is}$. Further, considering the independent increment property of the Wiener process, we can define the increment of degradation and time as Δx_{isj} and Δt, where $\Delta x_{isj} = x_{isj} - x_{is(j-1)}$ and $\Delta t_{isj} = t_{isj} - t_{is(j-1)}$. Here, we first assume α_i is observable for unit i, and set $\Omega = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. The specific steps will be illustrated using the iteration at the $l+1$th step as an example. Let the model parameters derived from the lth iteration are $\Theta^{(l)} = \{\mu_1^{(l)}, \mu_2^{(l)}, \sigma_1^{2(l)}, \sigma_2^{2(l)}, \sigma_\alpha^{(l)}\}$, and next we will carry out the $l+1$th iteration.

E-step: calculate the expected log-likelihood function $Q(\Theta|\boldsymbol{X}, \Theta^{(l)})$.

$$\begin{aligned} &Q(\Theta|\boldsymbol{X}, \Theta^{(l)}) = E_{\Omega|\Theta^{(l)}} L(\Theta|\boldsymbol{X}, \Omega) \\ &= C + \sum_{i=1}^{m} \left\{ \exp\left\{ \sigma_\alpha \Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right) \right\} - \frac{E(\alpha_i^2) - 2\mu_\alpha E(\alpha_i) + \mu_\alpha^2}{2\sigma_\alpha^2} \right\} \\ &- \sum_{i=1}^{m} \sum_{s=1}^{2} \sum_{j=1}^{n_{is}} \left[\frac{1}{2} \ln \sigma_s^2 + \frac{1}{2} \ln(\Delta t_{isj}) \right. \\ &\left. + \frac{(\Delta x_{isj})^2 - 2\Delta x_{isj} \mu_s \Delta t_{isj} E(\alpha_i) + \mu_s^2 (\Delta t_{isj})^2 E(\alpha_i^2)}{2\sigma_s^2 \Delta t_{isj}} \right] \end{aligned} \tag{4}$$

where C is a constant, $E(\alpha_i)$ and $E(\alpha_i^2)$ represent the expectation of α_i and α_i^2.

From the Eq.(4), it can be seen that we need to obtain the values of $E(\alpha_i)$ and $E(\alpha_i^2)$. Using statistical knowledge, $E(\alpha_i^2)$ can be calculated through $E(\alpha_i^2) = (E(\alpha_i))^2 + D(\alpha_i)$, where $D(\alpha_i)$ is the variance of α_i. Therefore, what we essentially need is to obtain the mean and variance of α_i, which can be inferred through statistical inference from the posterior distribution of α_i. Using the parameter estimates from the lth iteration, the prior distribution of α_i for the $l+1$th iteration can be directly obtained. Further, by combining the observed data, the likelihood function can be written, and thus, the posterior distribution of α_i can be obtained using Bayes' theorem. The detailed process can be written as

$$p(\alpha_i|\boldsymbol{X}, \Theta^{(l)}) \propto p(\boldsymbol{X}|\alpha_i, \Theta^{(l)}) p(\alpha_i|\Theta^{(l)}) \tag{5}$$

From Eq.(5), it can be obtained that the posterior distribution of α_i remains the truncated normal distribution, denoted as $\alpha_i \sim TN(\mu_{\alpha i}^{(l+1)}, \sigma_{\alpha i}^{2(l+1)})$

$$\begin{aligned}\sigma_{\alpha i}^{2(l)} &= \left[\left(\sigma_{\alpha}^{2(l)}\right)^{-1} + \sum_{s=1}^{2}\left(\sigma_{s}^{2(l)}\right)^{-1}\left(\mu_{s}^{(l)}\right)^{2} t_{isn_{is}}\right]^{-1} \\ \mu_{\alpha i}^{(l)} &= \sigma_{\alpha i}^{2(l)}\left[\mu_{\alpha}^{(l)}\left(\sigma_{\alpha}^{2(l)}\right)^{-1} + \sum_{s=1}^{2}\left(\sigma_{s}^{2(l)}\right)^{-1}\mu_{s}^{(l)} x_{isn_{is}}\right]\end{aligned} \tag{6}$$

At this point, the complete expression for $Q(\Theta|\boldsymbol{X}, \Theta^{(l)})$ can be provided, and then it need to be maximized. Next, we can perform the M-step.

M-step: maximizing the expected log-likelihood function. From Eq.(4), the expected log-likelihood function $Q(\Theta|\boldsymbol{X}, \Theta^{(l)})$ can be divided into two unrelated parts, denoted as $Q_1(\Theta_1|\boldsymbol{X}, \Theta^{(l)})$ and $Q_2(\sigma_\alpha|\boldsymbol{X}, \Theta^{(l)})$, which can be maximized separately, where $\Theta_1 = \{\mu_1, \mu_2, \sigma_1^2, \sigma_2^2\}$. Their specific expressions can be written as

$$\begin{aligned}Q_1(\Theta_1|\boldsymbol{X}, \Theta^{(l)}) &= -\sum_{i=1}^{m}\sum_{s=1}^{2}\sum_{j=1}^{n_{is}}\left[\frac{1}{2}\ln\sigma_s^2 + \frac{1}{2}\ln\left(\Delta t_{isj}\right)\right. \\ &\left. + \frac{(\Delta x_{isj})^2 - 2\Delta x_{isj}\mu_s\Delta t_{isj}E(\alpha_i) + \mu_s^2\left(\Delta t_{isj}\right)^2 E(\alpha_i^2)}{2\sigma_s^2\Delta t_{isj}}\right] \\ Q_2(\sigma_\alpha|\boldsymbol{X}, \Theta^{(l)}) &= \sum_{i=1}^{m}\left\{\exp\left\{\sigma_\alpha\Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)\right\}\right. \\ &\left. - \frac{E(\alpha_i^2) - 2\mu_\alpha E(\alpha_i) + \mu_\alpha^2}{2\sigma_\alpha^2}\right\}\end{aligned} \tag{7}$$

For $Q_1(\Theta_1|\boldsymbol{X}, \Theta^{(l)})$, we can obtain the estimate of Θ_1 by setting its first derivative with respect to the unknown parameters to zero. Then we can express the analytical estimates of the unknown parameters for the $l + 1$th iteration as

$$\begin{aligned}\mu_s^{(l+1)} &= \frac{\sum_{i=1}^{m} E\left(\alpha_i\right) x_{isn_{is}}}{\sum_{i=1}^{m} E(\alpha_i^2) t_{isn_{is}}} \\ \sigma_s^{2(l+1)} &= \frac{1}{\sum_{i=1}^{m} n_{is}}\sum_{i=1}^{m}\sum_{j=1}^{n_{is}}\left[\frac{\left(\Delta x_{isj}\right)^2}{\Delta t_{isj}}\right. \\ &\left. -2\Delta x_{isj}\mu_s^{(l+1)}E\left(\alpha_i\right) + \left(\mu_s^{(l+1)}\right)^2\Delta t_{isj}E(\alpha_i^2)\right]\end{aligned} \tag{8}$$

where $s = 1, 2$.

For $Q_2(\sigma_\alpha|\boldsymbol{X}, \Theta^{(l)})$, its expression is more complex, making it impossible to obtain an analytical estimate for σ_α in the same way as maximizing $Q_1(\Theta_1|\boldsymbol{X}, \Theta^{(l)})$. Therefore, a numerical optimization algorithm is needed to solve this. Specifically, we employ the "fminbnd" function in MATLAB.

By following the above steps, $\Theta^{(l+1)}$ can be obtained. We can then continue the iteration until the change in parameters reaches a predetermined threshold, indicating that the algorithm has converged. Specifically, the EM algorithm is considered to have converged when the absolute values of the relative changes of all parameters are less than 10^{-6}. The parameters obtained in this iteration will serve as the final estimates.

Additionally, it is important to note that the algorithm requires initialization of the unknown parameters at the beginning, meaning that initial values for the unknown parameters must be provided. The appropriateness of initial values can significantly impact both the accuracy and efficiency of the algorithm. Therefore, to construct effective initial guesses that can avoid slow convergence or local optima far from the true MLE, we propose a four-step method. The specific steps are as follows.

1) Calculate the degradation rates μ_{is} of the two PCs for all units through linear fitting, $i = 1, 2, \ldots, m$, $s = 1, 2$, the specific calculation process can be provided as

$$\mu_{is} = \frac{x_{isn_{is}}}{t_{isn_{is}}} \tag{9}$$

2) Obtain the initial guess of $\{\mu_1, \mu_2\}$ by calculating the mean of μ_{is}

$$\hat{\mu}_s = \frac{\mu_{is}}{m} \tag{10}$$

3) Get the initial value of $\{\sigma_1^2, \sigma_2^2\}$ by MLE

$$L_{1s} = \prod_{i=1}^{m}\prod_{j=1}^{n_{is}}\frac{1}{\sqrt{2\pi\sigma_s^2\Delta t_{isj}}}\exp\left\{-\frac{\left(\Delta x_{isj} - \mu_{is}\Delta t_{isj}\right)^2}{2\sigma_s^2\Delta t_{isj}}\right\} \tag{11}$$

Then the expression of $\hat{\sigma}_s^2$ can be derived as

$$\hat{\sigma}_s^2 = \frac{1}{\sum_{i=1}^{m} n_{is}}\sum_{i=1}^{m}\sum_{j=1}^{n_{is}}\left[\frac{\left(\Delta x_{isj}\right)^2}{\Delta t_{isj}} - 2\Delta x_{isj}\mu_{is} + \mu_{is}^2\Delta t_{isj}\right] \tag{12}$$

4) Obtain the initial guess of σ_α by MLE

$$L_2 = \prod_{i=1}^{m}\frac{\phi\left(\frac{\alpha_i - \mu_\alpha}{\sigma_\alpha}\right)}{\sigma_\alpha\Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)} \tag{13}$$

where α_i can be calculated as

$$\alpha_i = \frac{\mu_{i1} + \mu_{i2}}{\mu_1 + \mu_2} \tag{14}$$

In the actual computation, the calculation of σ_α needs to be carried out using a simple numerical optimization algorithm. In this study, we implement it using the "fminbnd" function in MATLAB.

3. RELIABILITY ANALYSIS

As a key issue in practical engineering, lifetime distribution under the conception of FHT are focused in this section. For a bivariate degradation situation, a product is considered to have failed when either PC reaches its failure threshold. We define the FHT;s lifetimes of the two PCs as T_1 and T_2. Consequently, the product's lifetime T is taken as the minimum of T_1 and T_2. The specific form of the product lifetime distribution $F_T(t)$ can be expressed by

$$F_T(t) = F_{T_1}(t) + F_{T_2}(t) - F_{T_1,T_2}(t,t) \tag{15}$$

where $F_{T_1}(t)$ and $F_{T_2}(t)$ are the lifetime time distributions of PC_1 and PC_2, respectively. $F_{T_1,T_2}(t,t)$ is the joint lifetime distribution of the two PCs. As discussed earlier, when α is a deterministic value, the two PCs are independent. This means that when α is known, $F_{T_1,T_2}(t,t)$ can be expressed as the product of $F_{T_1}(t)$ and $F_{T_2}(t)$. Therefore, the conditional lifetime distribution of the product with respect to α can be expressed as

$$\begin{aligned} F_{T|\alpha}(t\,|\alpha) &= F_{T_1|\alpha}(t_1\,|\alpha) + F_{T_2|\alpha}(t_2\,|\alpha) \\ &\quad - F_{T_1|\alpha}(t_1\,|\alpha)\,F_{T_2|\alpha}(t_2\,|\alpha) \end{aligned} \tag{16}$$

where $F_{T_1|\alpha}(t_1\,|\alpha)$ and $F_{T_2|\alpha}(t_2\,|\alpha)$ are the conditional lifetime distribution of PC_1 and PC_2 with respect to α, respectively. It is well known that, for a linear Wiener process, the FHT's lifetime distribution follows an inverse Gaussian distribution. As a result, these two lifetime distributions can be written as

$$\begin{aligned} F_{T_s|\alpha}(t_s\,|\alpha) &= \Phi\left(\frac{-\omega_s + \alpha\mu_s t_s}{\sigma_s\sqrt{t_s}}\right) \\ &\quad + \exp\left\{\frac{2\alpha\mu_s\omega_s}{\sigma_s^2}\right\}\Phi\left(\frac{-\omega_s - \alpha\mu_s t_s}{\sigma_s\sqrt{t_s}}\right), s = 1,2 \end{aligned} \tag{17}$$

where ω_1 and ω_2 are the failure thresholds of the PC_1 and PC_2, respectively. However, considering the model proposed in this paper, α is a random variable. Therefore, in order to obtain the complete CDF of T, we need to integrate $F_{T|\alpha}(t\,|\alpha)$ with respect to α. Essentially, we need to derive the expectation of $F_{T|\alpha}(t\,|\alpha)$ with respect to α. Therefore, the unconditional CDF of T can be represented as

$$\begin{aligned} F_T(t) &= E_\alpha\left(F_{T_1|\alpha}(t\,|\alpha)\right) + E_\alpha\left(F_{T_2|\alpha}(t\,|\alpha)\right) \\ &\quad - E_\alpha\left(F_{T_1|\alpha}(t\,|\alpha)\,F_{T_2|\alpha}(t\,|\alpha)\right) \end{aligned} \tag{18}$$

where $E_\alpha\left(F_{T_1|\alpha}(t\,|\alpha)\right)$ and $E_\alpha\left(F_{T_2|\alpha}(t\,|\alpha)\right)$ are the unconditional CDFs of T_1 and T_2, respectively. $E_\alpha\left(F_{T_1|\alpha}(t\,|\alpha)\,F_{T_2|\alpha}(t\,|\alpha)\right)$ is the unconditional joint CDF of T_1 and T_2.

Building upon Theorem 4 in (Pan, Liu, Huang, Cao, & Alsaedi, 2017), we further extend it to the shared frailty factor model proposed in this paper, and the specific expressions for $E_\alpha\left(F_{T_s|\alpha}(t\,|\alpha)\right)$ can be given as

$$\begin{aligned} E_\alpha\left(F_{T_s|\alpha}(t\,|\alpha)\right) = \Phi^{-1}\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)\Bigg\{&\Phi\left(\frac{-\omega_s + \mu_\alpha\mu_s t}{\sqrt{\sigma_s^2 t + \sigma_\alpha^2\mu_s^2 t^2}}\right) - \Phi_2\left(\frac{-\omega_s + \mu_\alpha\mu_s t}{\sqrt{\sigma_s^2 t + \sigma_\alpha^2\mu_s^2 t^2}}, -\frac{\mu_\alpha}{\sigma_\alpha}, -\frac{\sigma_\alpha\mu_s\sqrt{t}}{\sqrt{\sigma_s^2 + \sigma_\alpha^2\mu_s^2 t}}\right) \\ &+ \exp\left(\frac{2\mu_\alpha\mu_s\omega_s}{\sigma_s^2} + \frac{2\sigma_\alpha^2\mu_s^2\omega_s^2}{\sigma_s^4}\right)\times\Bigg[\Phi\left(-\frac{\omega_s\sigma_s^2 + \mu_\alpha\mu_s\sigma_s^2 t + 2\omega_s\sigma_\alpha^2\mu_s^2 t}{\sigma_s^2\sqrt{\sigma_s^2 t + \sigma_\alpha^2\mu_s^2 t^2}}\right) \\ &-\Phi_2\left(-\frac{\omega_s\sigma_s^2 + \mu_\alpha\mu_s\sigma_s^2 t + 2\omega_s\sigma_\alpha^2\mu_s^2 t}{\sigma_s^2\sqrt{\sigma_s^2 t + \sigma_\alpha^2\mu_s^2 t^2}}, -\frac{\mu_\alpha}{\sigma_\alpha} - \frac{2\sigma_\alpha\mu_s\omega_s}{\sigma_s^2}, \frac{\sigma_\alpha\mu_s\sqrt{t}}{\sqrt{\sigma_s^2 + \sigma_\theta^2\mu_s^2 t}}\right)\Bigg]\Bigg\} \end{aligned} \tag{19}$$

where $s = 1, 2$.

Next, we need to provide the specific expression for the other part of $F_T(t)$, namely the joint lifetime distribution of the two PCs, which are $E_\alpha\left(F_{T_1|\alpha}(t\,|\alpha)\,F_{T_2|\alpha}(t\,|\alpha)\right)$. Compared to existing research on single PC degradation considering random effects, the dimensionality of the joint lifetime distribution increases, making the expression of the conditional lifetime distribution with integrals more complex. Simultaneously, in contrast to the current studies where the shared frailty factor is assumed to follow a normal distribution in bivariate degradation modeling, the assumption of a truncated normal distribution introduces non-complete real-number limits for the integral bounds, thus necessitating further consideration of the truncation effect. Overall, the conclusions derived from existing studies on FHT's lifetime distributions do not apply to the model proposed in this paper. Therefore, to derive the explicit expression for $E_\alpha\left(F_{T_1|\alpha}(t\,|\alpha)\,F_{T_2|\alpha}(t\,|\alpha)\right)$, we introduce Lemma 1, Proposition 1, and Proposition 2.

Lemma 1 ((Owen, 1981)). if $Z \sim N(\mu, \sigma^2)$ and $a_1, a_2, a_3, a_4 \in$ R, then the following holds

$$\int_{-\infty}^{y} \Phi\left(a_1 + a_2 z\right) \Phi\left(a_3 + a_4 z\right) \phi(z) dx = tvn\left[\frac{a_3}{\sqrt{1+a_4^2}}, \frac{a_1}{\sqrt{1+a_2^2}}, y; \frac{a_2 a_4}{\sqrt{\left(1+a_2^2\right)\left(1+a_2^4\right)}}, -\frac{a_2}{\sqrt{1+a_2^2}}, -\frac{a_4}{\sqrt{1+a_4^2}}\right] \tag{20}$$

where $tvn(\cdot)$ is the cumulative distribution function of a three-dimensional normal distribution.

Proposition 1. if $Z \sim TN(\mu, \sigma^2)$ and $A_1, A_2, A_3, A_4 \in$ R, then the following holds

$$\begin{aligned} E_Z\left(\Phi\left(A_1 + A_2 Z\right) \Phi\left(A_3 + A_4 Z\right)\right) = \frac{1}{\Phi\left(\frac{\mu}{\sigma}\right)} & \left[bvn\left(\frac{A_1 + A_2\mu}{\sqrt{1+\left(A_2\sigma\right)^2}}, \frac{A_3 + A_4\mu}{\sqrt{1+\left(A_4\sigma\right)^2}}; \frac{A_2 A_4 \sigma^2}{\sqrt{1+\left(A_2\sigma\right)^2}\sqrt{1+\left(A_4\sigma\right)^2}}\right)\right. \\ & \left. -tvn\left(\frac{A_1 + A_2\mu}{\sqrt{1+\left(A_2\sigma\right)^2}}, \frac{A_3 + A_4\mu}{\sqrt{1+\left(A_4\sigma\right)^2}}, -\frac{\mu}{\sigma}; \frac{A_2 A_4 \sigma^2}{\sqrt{1+\left(A_2\sigma\right)^2}\sqrt{1+\left(A_4\sigma\right)^2}}, -\frac{A_2\sigma}{\sqrt{1+\left(A_2\sigma\right)^2}}, -\frac{A_4\sigma}{\sqrt{1+\left(A_4\sigma\right)^2}}\right)\right] \end{aligned} \tag{21}$$

where $bvn(\cdot)$ is the cumulative distribution function of a two-dimensional normal distribution.

Proposition 2. if $Z \sim TN(\mu, \sigma^2)$ and $B_1, B_2, B_3, B_4, B_5 \in$ R, then the following holds

$$\begin{aligned} & E_Z\left(\exp\left(B_5 Z\right) \Phi\left(B_1 + B_2 Z\right) \Phi\left(B_3 + B_4 Z\right)\right) = \frac{1}{\Phi\left(\frac{\mu}{\sigma}\right)} \exp\left(B_5\mu + \frac{B_5^2\sigma^2}{2}\right) \\ & \times \left[bvn\left(\frac{B_1 + B_2\mu + B_2 B_5\sigma^2 + B_2\sigma\mu}{\sqrt{1+\left(B_2\sigma\right)^2}}, \frac{B_3 + B_4\mu + B_4 B_5\sigma^2 + B_4\sigma\mu}{\sqrt{1+\left(B_4\sigma\right)^2}}; \frac{B_2 B_4\sigma^2}{\sqrt{1+\left(B_2\sigma\right)^2}\sqrt{1+\left(B_4\sigma\right)^2}}\right)\right. \\ & -tvn\left(\frac{B_1 + B_2\mu + B_2 B_5\sigma^2 + B_2\sigma\mu}{\sqrt{1+\left(B_2\sigma\right)^2}}, \frac{B_3 + B_4\mu + B_4 B_5\sigma^2 + B_4\sigma\mu}{\sqrt{1+\left(B_4\sigma\right)^2}}, -\frac{\mu}{\sigma} - B_5\sigma; \frac{B_2 B_4\sigma^2}{\sqrt{1+\left(B_2\sigma\right)^2}\sqrt{1+\left(B_4\sigma\right)^2}},\right. \\ & \left.\left. -\frac{B_2\sigma}{\sqrt{1+\left(B_2\sigma\right)^2}}, -\frac{B_4\sigma}{\sqrt{1+\left(B_4\sigma\right)^2}}\right)\right] \end{aligned} \tag{22}$$

The proof of the above propositions can be found in Appendix. Based on Proposition 1 and Proposition 2, the specific expression of $F_{T_1,T_2}(t,t)$ can be given as

$$F_{T_1,T_2}(t,t) = U_1 + U_2 + U_3 + U_4 \tag{23}$$

where

$$\begin{aligned} U_1 = \int_0^{+\infty} \Phi\left(\frac{-\omega_1 + \alpha\mu_1 t}{\sigma_1\sqrt{t}}\right) \Phi\left(\frac{-\omega_2 + \alpha\mu_2 t}{\sigma_2\sqrt{t}}\right) f(\alpha) d\alpha = \left(\Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)\right)^{-1} & \left[bvn\left(\frac{-\omega_1 + \mu_1\omega_1\mu_\alpha}{p_3}, \frac{-\omega_2 + \mu_2\omega_2\mu_\alpha}{p_4}; \frac{p_1 p_2}{p_3 p_4}\right)\right. \\ & \left. -tvn\left(\frac{-\omega_1 + \mu_1\omega_1\mu_\alpha}{p_3}, \frac{-\omega_2 + \mu_2\omega_2\mu_\alpha}{p_4}, -\frac{\mu_\alpha}{\sigma_\alpha}; \frac{p_1 p_2}{p_3 p_4}, -\frac{p_1}{p_3}, -\frac{p_2}{p_4}\right)\right] \end{aligned}$$

$$\begin{aligned} U_2 = \int_0^{+\infty} \exp\left\{\frac{2\alpha\mu_2\omega_2}{\sigma_2^2}\right\} \Phi\left(\frac{-\omega_1 + \alpha\mu_1 t}{\sigma_1\sqrt{t}}\right) \Phi\left(\frac{-\omega_2 - \alpha\mu_2 t}{\sigma_2\sqrt{t}}\right) f(\alpha) d\alpha = \left(\Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)\right)^{-1} \exp\left(\frac{2\mu_2\omega_2\mu_\alpha}{\sigma_2^2} + \frac{2p_6^2}{\sigma_2^4}\right) \\ \times \left[bvn\left(\frac{-\omega_1 + \mu_1\omega_1\mu_\alpha + 2p_1 p_6 \sigma_2^{-2} + p_1\mu_\alpha}{p_3}, \frac{-\omega_2 - \mu_2\omega_2\mu_\alpha - 2p_2 p_6\sigma_2^{-2} - p_2\mu_\alpha}{p_4}; -\frac{p_1 p_2}{p_3 p_4}\right)\right. \\ \left. -tvn\left(\frac{-\omega_1 + \mu_1\omega_1\mu_\alpha + 2p_1 p_6\sigma_2^{-2} + p_1\mu_\alpha}{p_3}, \frac{-\omega_2 - \mu_2\omega_2\mu_\alpha - 2p_2 p_6\sigma_2^{-2} - p_2\mu_\alpha}{p_4}, -\frac{\mu_\alpha}{\sigma_\alpha} - \frac{2p_6}{\sigma_2^2}; -\frac{p_1 p_2}{p_3 p_4}, -\frac{p_1}{p_3}, \frac{p_2}{p_4}\right)\right] \end{aligned}$$

$$U_3 = \int_0^{+\infty} \exp\left\{\frac{2\alpha\mu_1\omega_1}{\sigma_1^2}\right\} \Phi\left(\frac{-\omega_1-\alpha\mu_1 t}{\sigma_1\sqrt{t}}\right) \Phi\left(\frac{-\omega_2+\alpha\mu_2 t}{\sigma_2\sqrt{t}}\right) f(\alpha)d\alpha = \left(\Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)\right)^{-1} \exp\left(\frac{2\mu_1\omega_1\mu_\alpha}{\sigma_1^2}+\frac{2p_5^2}{\sigma_1^4}\right)$$
$$\times \Bigg[bvn\left(\frac{-\omega_1-\mu_1\omega_1\mu_\alpha-2p_1p_5\sigma_1^{-2}-p_1\mu_\alpha}{p_3}, \frac{-\omega_2+\mu_2\omega_2\mu_\alpha+2p_2p_5\sigma_1^{-2}+p_2\mu_\alpha}{p_4}; -\frac{p_1p_2}{p_3p_4}\right)$$
$$-tvn\left(\frac{-\omega_1-\mu_1\omega_1\mu_\alpha-2p_1p_5\sigma_1^{-2}-p_1\mu_\alpha}{p_3}, \frac{-\omega_2+\mu_2\omega_2\mu_\alpha+2p_2p_5\sigma_1^{-2}+p_2\mu_\alpha}{p_4}, -\frac{\mu_\alpha}{\sigma_\alpha}-\frac{2p_5}{\sigma_1^2}; -\frac{p_1p_2}{p_3p_4}, \frac{p_1}{p_3}, -\frac{p_2}{p_4}\right)\Bigg]$$

$$U_4 = \int_0^{+\infty} \exp\left\{\frac{2\alpha\mu_1\omega_1}{\sigma_1^2}+\frac{2\alpha\mu_2\omega_2}{\sigma_2^2}\right\} \Phi\left(\frac{-\omega_1-\alpha\mu_1 t}{\sigma_1\sqrt{t}}\right) \Phi\left(\frac{-\omega_2-\alpha\mu_2 t}{\sigma_2\sqrt{t}}\right) f(\alpha)d\alpha$$
$$= \left(\Phi\left(\frac{\mu_\alpha}{\sigma_\alpha}\right)\right)^{-1} \exp\left(\left(\frac{2\mu_1\omega_1}{\sigma_1^2}+\frac{2\mu_2\omega_2}{\sigma_2^2}\right)\mu_\alpha + \frac{2\left(p_5^2\sigma_2^4+2p_5p_6\sigma_1^2\sigma_2^2+p_6^2\sigma_1^4\right)}{\sigma_1^4\sigma_2^4}\right)$$
$$\Bigg[bvn\left(\frac{-\omega_1-\mu_1\omega_1\mu_\alpha-2p_1p_5\sigma_2^{-2}-2p_1p_6\sigma_1^{-2}-p_1\mu_\alpha}{p_3}, \frac{-\omega_2-\mu_2\omega_2\mu_\alpha-2p_2p_5\sigma_2^{-2}-2p_2p_6\sigma_2^{-2}-p_2\mu_\alpha}{p_4}; \frac{p_1p_2}{p_3p_4}\right)$$
$$-tvn\left(\frac{-\omega_1-\mu_1 D_1\mu_\alpha-2p_1p_5\sigma_2^{-2}-2p_1p_6\sigma_1^{-2}-p_1\mu_\alpha}{p_3}, \frac{-\omega_2-\mu_2\omega_2\mu_\alpha-2p_2p_5\sigma_2^{-2}-2p_2p_6\sigma_2^{-2}-p_2\mu_\alpha}{p_4},\right.$$
$$\left.-\frac{\mu_\alpha}{\sigma_\alpha}-\frac{2p_5}{\sigma_1^2}-\frac{2p_6}{\sigma_2^2}; \frac{p_1p_2}{p_3p_4}, \frac{p_1}{p_3}, \frac{p_2}{p_4}\right)\Bigg]$$

where

$$p_1 = \mu_1\sigma_\alpha t \qquad p_3 = \sqrt{\sigma_1^2 t + \mu_1^2\sigma_\alpha^2 t^2} \qquad p_5 = \mu_1\omega_1\sigma_\alpha$$
$$p_2 = \mu_2\sigma_\alpha t \qquad p_4 = \sqrt{\sigma_2^2 t + \mu_2^2\sigma_\alpha^2 t^2} \qquad p_6 = \mu_2\omega_2\sigma_\alpha$$

Therefore, combining Eq.(19) and Eq.(23), we can obtain $F_T(t)$. And then, we have completed the reliability analysis.

4. CASES STUDY

In this section, we provide a numerical simulation example and a wheel wear case study to demonstrate the validity and efficacy of the proposed model and methods. First, in the numerical simulation example, we verify the effectiveness of the parameter estimation and reliability analysis methods. Furthermore, we showcase the advantage of our model through a case study of wheel wear.

4.1. Numerical simulation example

Initially, a numerical simulation example is conducted to validate the performance of the proposed parameter estimation and reliability analysis methods. The unknown model parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_\alpha)$ are set to (1.2, 1, 0.16, 0.09, 0.6). The simulation approach is as follows: first, the vulnerability factor α_i for the ith unit is generated from the truncated normal distribution $TN(\mu_\alpha, \sigma_\alpha^2)$, $i = 1, 2, \ldots, m$. Next, the degradation processes are generated from the perspective of independent increments, leveraging the conditional independence of $X_1(t)|\alpha$ and $X_2(t)|\alpha$. We consider that the two PCs are measured simultaneously, and the measurement interval Δt is defined as 0.1, which means that measurements are taken at time $t = 0.1j$, $j = 1, 2, \ldots, n$. Here, m and n represent the number of units and measurements, respectively. To thoroughly evaluate the performance of the proposed parameter estimation method, we tested its stability and accuracy under different sample size settings. Therefore, we consider different combinations of sample sizes and measurement counts, specifically setting m to (5, 15) and n to (10, 20). For each condition, we perform the simulation 1,000 times to obtain the mean and root mean square error(RMSE) of the estimated parameters. The results are presented in Table 1. The parameter estimation results indicate that, regardless of whether the unit size m is small or large, the mean of the estimates is very close to the true value. Additionally, as the number of measurements increases, the RMSE results show an improvement in the precision of the parameter estimates, which aligns with objective trends. These conclusions validate the effectiveness of the proposed parameter estimation method.

Furthermore, we set the failure thresholds of the two PCs, ω_1 and ω_2, to 5 and 4, respectively. Using the proposed analytical reliability analysis method, we obtained the CDF curve of

Table 1. Mean and RMSE of the parameters estimates based on 1000 replications.

(m, n)	Indices	μ_1	μ_2	σ_1^2	σ_2^2	σ_α
(5,10)	Mean	1.1993	1.0090	0.1583	0.0890	0.5652
	RMSE	0.3276	0.2717	0.0349	0.0195	0.5008
(5,20)	Mean	1.2040	1.0051	0.1581	0.0899	0.5680
	RMSE	0.3062	0.2550	0.0233	0.0131	0.4282
(15,10)	Mean	1.2017	0.9974	0.1591	0.0896	0.5987
	RMSE	0.1884	0.1575	0.0201	0.0114	0.2838
(15,20)	Mean	1.1993	1.0035	0.1595	0.0901	0.5981
	RMSE	0.1805	0.1457	0.0135	0.0079	0.2288

the product's lifetime under the concept of FHT. We also included the curves obtained through Monte Carlo simulations as the benchmark for comparison. Specifically, based on the true values of the model parameters, we simulate the product degradation path using the independent increment property of the Wiener process to obtain the product's lifetime. This process is repeated multiple times, and the lifetimes are statistically analyzed to obtain the CDF curve. The results are shown in Fig.1. It can be observed that the lifetime distribution curves obtained from both methods(the proposed analytical reliability analysis method and the Monte Carlo numerical simulation method) show very little difference, which verifies the effectiveness of the reliability analysis method proposed in this paper.

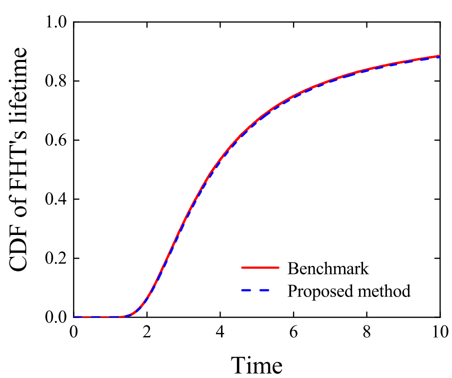

Figure 1. the CDF curve of the FHT's lifetime of the simulation example.

4.2. Real application

In this section, a practical engineering example is provided. For trains, wheel wear is one of the primary factors leading to a gradual decline in performance. When the wear reaches a certain fixed value, it is considered that the train has failed and can no longer operate. Therefore, the wear amount on the wheel diameter can be taken as an PC of train performance. Freitas et al.(Freitas, de Toledo, Colosimo, & Pires, 2009) published a dataset depicting wear of a specific wheel from 14 trains continuously monitored by a railway company in Brazil. Following the common data processing method adopted by current existing bivariate degradation studies, the 14 wheel samples are divided into two groups, each treated as a PC, as shown in Fig.2. Actually, we consider a practical scenario in which a train has two critical wheels. The degradation of each wheel can be regarded as a PC, and the failure of either wheel would result in the failure of the entire train. Since the two wheels of the same train experience the same environment, load, and operating conditions, their failure processes exhibit interdependence. The degradation measurement is the wear value of wheel diameter, and it is defined that the wheel is considered to have failed when the wear reaches 77 mm.

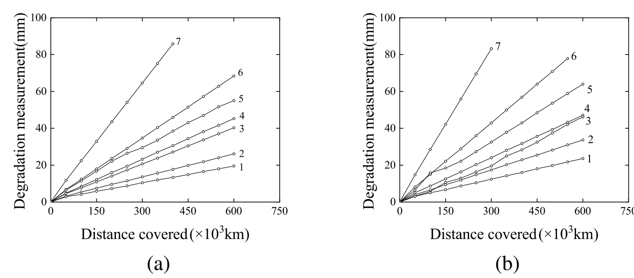

Figure 2. Divided wheel wear data.(a) PC_1, (b) PC_2.

The proposed model is defined as M_0 and adopted to fit the degradation data in Fig.2. And model parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_\alpha)$ are estimated as (0.0908, 0.1115, 0.0052, 0.0136, 0.8247). Model M_1 defined in Section 4.1(the shared frailty factor follows a normal distribution) is applied as a reference, AIC values 360.65 for M_0 and 362.12 for M_1 are obtained to demonstrate the fitting goodness. Clearly, M_0 illustrates a better fitting goodness.

In addition, comparative results of life distribution CDF curve obtained from M_0 and M_1 are shown in Fig.3. It is evident that there are noticeable differences between the reliability analysis results estimated by the two models. Specifically, the CDF of product lifetime describes the probability that the product will fail at or before a specific time, serving as an important indicator for assessing the reliability of a product.

From Fig.3, it can be seen that when the product failure probability is low, M_1 provides a more optimistic estimate of the product's lifetime compared to M_0, suggesting a longer product lifetime under the same failure probability. This may lead to potential risks in practical engineering due to the failure to perform timely maintenance, highlighting the need for proactive maintenance. On the other hand, as the failure probability of the product increases, M_1 offers a more conservative estimate, which may lead to unnecessary waste of maintenance resources. In this case, the frequency of preventive maintenance can be reduced.

Furthermore, to evaluate the goodness-of-fit, the empirical CDF curve is obtained using the Kaplan–Meier method, and the results are shown in Fig. 3. It can be observed that the CDF estimated based on model M_0 exhibits good agreement with the empirical CDF. From a quantitative perspective, the goodness-of-fit of the two models was compared by calculating the area difference between the estimated CDF and the empirical CDF. The area differences for models M_0 and M_1 were 620.61 and 847.13, respectively. As the area difference of model M_0 is smaller, it indicates that model M_0 provides a more accurate prediction of the lifetime distribution.

For a better understanding, we provided the estimated values for the two quantile lifetimes, $t_{0.1}$ and $t_{0.5}$, as shown in Table 2. The results indicate that inaccuracies in model estimation can lead to significant biases in the reliability estimation results. As mentioned earlier, when the product's failure probability is low, such as $p = 0.1$, M_1 provides a more optimistic estimate, with the estimated $t_{0.1}$ being larger. Simultaneously, when the failure probability increases, such as $p = 0.5$, M_1 offers a more conservative estimate, with $t_{0.5}$ being shorter. In both cases, this could lead to the formulation of unreasonable maintenance strategies, resulting in either the occurrence of risks or the waste of resources.

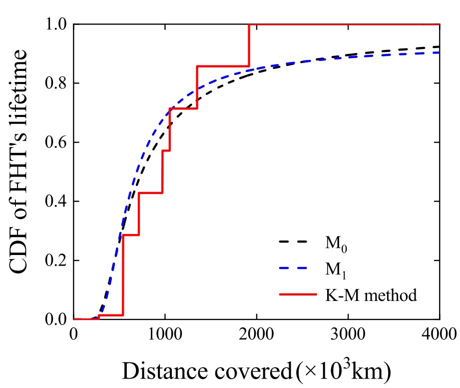

Figure 3. the estimated product FHT's lifetime distribution CDF curves

Table 2. Estimation of $t_{0.1}$ and $t_{0.5}$ for the train wear failure using M_0 and M_1 .

Model	$t_{0.1}(\times 10^3\text{km})$	$t_{0.5}(\times 10^3\text{km})$
M_0	367.52	748.98
M_1	380.24	690.59

5. CONCLUSION

This paper aims to investigate the issues of bivariate degradation modeling and reliability analysis. Considering the physical background where the degradation rates of PCs are always positive or negative during the actual degradation process, we propose a bivariate degradation model based on Wiener processes and a shared frailty factor with the truncated normal distribution. Furthermore, a specific estimation method for the unknown model parameters is provided based on the EM algorithm. Building on the existing normal distribution integral theory, we present the propositions for the bivariate truncated normal distribution integral, and further provide the analytical expressions for the CDF of product lifetime under the FHT framework.

Next, a numerical simulation example is used to validate the effectiveness of the proposed parameter estimation method and the correctness of the analytical expression for the FHT lifetime distribution, demonstrating excellent performance even under the small sample condition. Furthermore, a case study on wheel wear is conducted to validate the practical significance of the proposed model.

In the future, we plan to consider commonly observed nonlinear degradation and sensor noise in engineering products, and to investigate the new challenges they pose for parameter estimation and reliability analysis. In addition, we recognize that maintenance is crucial for ensuring product reliability, and thus we also aim to develop corresponding maintenance decision-making methods.

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