Learning Data-Driven Model of Damped Coupled Oscillators from System Impulse Response

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ABSTRACT
Data-driven modeling of dynamical systems has drawn much research attention recently, with the goal of approximating the underlying governing rules of a dynamical system with data-driven differential equations. Many real-world dynamical systems can be modeled as coupled oscillators, such as chemical reactors, ecological systems, integrated circuits, and mechanical systems. In those systems, the response can be described by coupled differential equations in which multi-dimensional state variables describe the time-evolution. However, in practice, the full set of state variables is often difficult or expensive to measure. This paper shows an attempt to develop a data-driven model for damped coupled oscillators from a mixed-mode response signal using neural differential equations. The univariate time-series data of the impulse response is first resampled into a multi-variate time-delayed embedding. A singular value decomposition is then applied to find the dominant orthogonal basis (oscillator modes). The decoupled modes are then modeled with parameterized neural differential equations. The unknown parameters can be learned from a segment of historical data. The proposed methodology is validated using impact testing data of an end mill in a machine tool spindle. The results demonstrate that the proposed method can effectively model damped coupled oscillators.

1. INTRODUCTION
Accurate modeling of dynamical systems is an important task towards health monitoring, prediction, and control. Traditionally, dynamical models are derived from known physical laws and the inherent properties of the systems. Such a dynamical model is typically represented by one or more differential equations, which can be used to model and simulate the system responses. The term ‘dynamical’ generally refers to the autonomous evolving process over time, governed by inherent system properties. In contrast, a ‘dynamic’ system can refer to any time-evolving system that either evolves autonomously or is driven by external forces. To model a dynamic system with an external force, we often first need to understand and model the dynamical behaviors of the system. Mathematically, many of the prediction problems of dynamical systems can be formulated as initial value problems. Furthermore, the goal of dynamical system modeling includes two parts: solving differential equations and identifying differential equations if unknown. For a simple human-designed system, an accurate physics-based dynamical model via differential equations may be constructed with high confidence. In that case, efficient solving of the differential equations is the main remaining task. However, for complex systems or in-situ systems, where the system parameters cannot be measured accurately, one would first need to identify the underlying differential equations. To achieve that, the missing information on the system properties can be determined with historical observation data. In practice, data-driven models or hybrid models can utilize the observed system data to help construct a dynamic model (Kutz et al., 2016).

Physics-informed machine learning is one of the most popular hybrid methods for the data-driven discovery of dynamic systems. Physics-informed neural networks (PINNs) have been proposed to learn and solve the underlying differential equations (Raissi et al., 2019). When the underlying physics is fully known, a neural network can be used as a numerical solver for either ordinary differential equations (ODEs) or partial differential equations (PDEs), as first reported by Lagaris et al. (Lagaris et al., 1998). In the case that the underlying differential equations are not fully known, a PINN can be adopted to learn the unknown coefficients in the differential equations with the help of historical data. The diagram given in Figure 1 summarizes how a PINN works with an ODE. The PINN assumes that the
solution to an ODE or a PDE can be approximated by a neural network. The solution is then constrained by the underlying physics-based differential equations. By minimizing the loss of the data and physics constraints, a data-driven solution and the unknown coefficients of the underlying ODE can be learned.

![Figure 1. Schematic of a PINN to learn a physics-informed ordinary differential equation.](image)

However, PINNs can only be applied when the forms of the underlying differential equations are fully defined, while the coefficients can be unknown. For complex and in-situ systems, one may not be able to fully determine an accurate form of the equations. In those cases, PINNs fail to learn and model the unknown systems. Intuitively, fully data-driven models can be built without any physics to model in-situ dynamic systems. In previous work, Fabro et al. built a regression model with a nonlinear function basis to represent a fully data-driven frequency response function in the frequency domain for machine spindles (Fabro et al., 2022). However, the model did not mimic physics in a physics-aware manner.

Recently, neural ordinary differential equations (NODEs) have been proposed to learn the underlying differential equations with no or little physics (Chen et al., 2018). In contrast to a PINN, a NODE constructs a parameterized differential equation and integrates an ODE solver into the model. Details of the learning process will be introduced in Section 2.1.2. NODEs provide a physics-aware approach to learning the underlying differential equations only from data. The term ‘physics-aware’ refers to the preservation of certain physical rules, such as a differential relationship. This research aims to investigate the feasibility of learning coupled differential equations from data.

Coupled oscillators are commonly found in practical engineering systems, such as bridges, buildings, geared systems, and machining spindles. Also, many in-situ systems can be modeled as coupled oscillators since there exists complex coupling among the sub-components. One of the characteristics of coupled oscillators is mixed modes in the measured system response. For example, a simple coupled oscillator with two springs and two masses will have two modes, one for each degree of freedom (DOF). Furthermore, each mode will have its own natural frequency. In practice, if the response for each DOF can be measured, the modes can be separated as an eigen-decomposition problem, even if the modes are coupled together in each measurement. If the response cannot be measured separately, then modal separation is needed. For coupled oscillators without damping, modal separation can be achieved with Fourier transforms to separate different frequency components. However, when damping is present, modal separation becomes more complicated as both the frequencies and associated time-decay terms must be separated. Therefore, the data-driven learning of damped coupled oscillators requires three steps: 1. Modal separation from data; 2. ODE learning for each mode; 3. Model synthesis. For linearly coupled systems, the model synthesis is just a simple sum of all modes.

Towards this end, this paper summarizes an attempt to investigate the feasibility of learning a data-driven model for damped coupled oscillators and use it to describe and predict the system dynamics. The rest of the paper is organized as follows: Section 2 presents the related background and the proposed learning method; Section 3 briefly introduces the experimental study to collect data and test the proposed method; Section 4 discusses the results and Section 5 concludes the paper.

2. METHODOLOGY

2.1. Related Background

In this section, we will introduce the existing methods for modal separation and ODE learning.

2.1.1 Mixed-mode decomposition

Modal separation of univariate mixed-mode signals has been extensively studied. Fourier transforms can effectively decompose a mixed-mode signal into various frequency components. In that case, each frequency component is represented with sinusoidal basis functions, so Fourier transforms are best suited for the analysis of steady-state responses. While Fourier transforms are effective and efficient for periodic signals, to approximate a non-periodic signal, an infinite number of Fourier basis functions are required, which is numerically inefficient. Independent component analysis (ICA) is one of the statistics-based methods for mixed-mode separation, which is widely used in the cocktail party problem or similar situations (De Lauro et al., 2005; Hyvärinen, 2013). ICA assumes the underlying processes are independent of each other and can handle non-Gaussian processes. However, ICA does not guarantee that the separated components are orthogonal, as with Fourier basis functions. Empirical mode decomposition (EMD) is another popular method for modal decomposition (Huang et al., 1998). Similar to ICA, EMD does not generate orthogonal modes, either.
Recently, Dylewsky et al. proposed a learning method named principal component trajectories to model spectrally continuous dynamics (Dylewsky et al., 2022). It was proved that by constructing a time-delay embedding for the time series and applying singular value decomposition (SVD) to the embedding data matrix, the principal component trajectories, which is a set of orthogonal basis functions, can be separated. While Dylewsky et al. modeled the underlying system as linear dynamics with dynamic mode decomposition (DMD), this work proposes to learn a neural ODE model for each mode and then assemble the models.

2.1.2 Neural Ordinary Differential Equations

As discussed above, NODEs provide a data-driven approach to learning the underlying differential equations. The learning process is summarized as follows. First, assume we have a differential equation in the form of Eq. 1.

\[
\dot{X} = f(X)
\]  

(1)

It was first proposed by Chen et al. that the function \( f(X) \) can be defined as a neural network with \( X \) as the inputs and \( \dot{X} \) as the outputs of the neural network (Chen et al., 2018). Then, the solution of this neural differential equation can be solved with an integration-based ODE solver, such as the Runge-Kutta fourth-order method. More advanced ODE solvers, such as Dopri5 (Dormand and Prince, 1980), can also be utilized in this process with varying step sizes as an attempt to obtain a more accurate solution, especially for irregular time intervals. Next, assume that the solution of the underlying differential equation can be written as a function of \( X \), more precisely as \( f(X(t)) \). With an ODE solver using the neural network as the differential function, and \( X(t_0) \) as the initial condition, \( X(t_0 + \Delta t) \) can be solved as:

\[
X(t_0 + \Delta t) = X(t_0) + \int_{t_0}^{t_0+\Delta t} X'(\tau)d\tau
\]  

(2)

Finally, the overall process of the neural ODE method can be written as:

\[
\dot{X}(t_0) = nn(X(t_0))
\]  

(3)

\[
X(t_1) = ODESolver(\dot{X}(t_0), X(t_0), t_0, t_1)
\]  

(4)

where \( nn(\cdot) \) represents a neural network.

The flowchart in Figure 2 shows the process of learning a neural differential equation to represent the system dynamics. If an example dataset in the form of a time series for \( X(t) \) can be collected, data pairs of \( X(t) \) and \( X(t + \Delta t) \) can be constructed as training data to train the data-driven model such that Eq. (2) can be satisfied. As a result, the optimized function \( f(X(t)) \) will represent a data-driven form of the underlying differential equation.

\[\begin{array}{c}
\text{Inputs} \quad f(X, \theta) \quad \text{Neural network outputs} \quad \text{ODE Solver} \\
\dot{X} = f(X, \theta) \\
\text{Model outputs} \quad [\dot{X}(t)]
\end{array}\]

Figure 2. Schematic of NODE.

2.2. Proposed Approaches

Based on the information presented in the introduction, a data-driven model for damped coupled oscillators is summarized in Figure 3. First, one can collect the impulse response from an in-situ dynamic system. Typically, a univariate mixed-mode response signal can be measured, denoted as \( X = [x_1, x_2, x_3, \ldots, x_n] \), where \( n \) is the number of samples.

Next, a time-delayed embedding can be constructed as:

\[
XX = \begin{bmatrix}
x_1 & x_2 & \cdots & x_{n-k+1} \\
x_2 & x_3 & \cdots & x_{n-k+2} \\
\vdots & \vdots & \ddots & \vdots \\
x_k & x_{k+1} & \cdots & x_n
\end{bmatrix}
\]  

(5)

where \( k \) is the number of embeddings. Then, a SVD can be performed on the embedding matrix as \( XX = U\Sigma V^T \). If we consider the columns as the spatial direction and the rows as the temporal direction, in an analogy to a spatial-temporal data matrix, the time-evolving trends will be captured by the columns of \( V \) after SVD. In the ideal scenario, each column of \( V \) will represent one temporal mode that can be modeled.
as an ODE function. For a spectrally-continuous system, we truncate $V$ to keep only the dominant modes. Each preserved mode is then learned separately as a NODE model. The NODE model can represent the dynamic response at any time. To predict the response, one can reconstruct the dynamic response with the simulated multiple-mode dynamics of $V_{\text{pred}}$ and the corresponding $U$ and truncated $\Sigma$. The corresponding $U$ and $\Sigma$ are constant assuming that the dynamical system is a linear time-invariant (LTI) system. This will allow us to use a small segment of the dynamical response to learn the system and extrapolate the full dynamic response.

3. Experimental Setup and Case Study

The discussions in Section 3 are organized as follows: Section 3.1 introduces the experimental setup and data collection process, and Section 3.2 presents the learning process with the collected response data from an end mill.

3.1. Experimental Setup and Data Collection

Figure 4 shows the experimental setup used to collect tool impact data on a machine tool.

![Experimental setup](image)

Figure 4: (a) Experimental setup within a machine tool and (b) close-up view of end mill and displacement sensor.

As seen in Figure 4a, an end mill with a diameter of 12.7 mm (0.5 in) was placed within a tool holder in a machine tool spindle, and an automated impactor was attached to the machine tool worktable. During impacting, a solenoid actuates the hammer of the impactor to cause an impact against the end mill, and a force sensor attached to the hammer measures the impact force while a fiber-optic displacement sensor measures the tool-to-workpiece displacement (Figure 4b). Furthermore, the fiber optic displacement sensor is mounted on a linear positioning stage with motion measured by a digital micrometer for sensor calibration purposes to convert voltage to displacement. For each of 50 impacts, force and displacement data were collected for two seconds at 51.2 kHz, with at least one second occurring after the impact. Figure 5 shows an example of the univariate (displacement) impulse response data. Because the displacement due to impacting is the tool-to-workpiece response, the displacement includes multiple modes that account for the entire structural loop within the machine tool. Hence, the system in Figure 4 represents a coupled mechanical system, and each impulse response, such as the example in Figure 5, is for mixed modes.

![Mixed-mode impulse response](image)

Figure 5: Mixed-mode impulse response of coupled mechanical system.

3.2. Learning Data-Driven Impulse Response Model of an End Mill

The univariate data is first embedded to a multivariate space to allow the extraction of oscillatory modes. After SVD was performed on the time-delayed embedding data, the first $r$ SVD components, columns of the $V$ matrix, were extracted. The first four modes are shown in Figure 6. It is worth noting that, in the decomposed singular vectors, $v_1$ and $v_2$ represent the first intrinsic mode, because $v_1$ and $v_2$ share the same frequency (1187.84 Hz) but differ with a phase shift of $\pi/2$ rad. This can be understood since SVD constructs two sinusoidal functions of the same frequency to represent one underlying mode with a phase shift, like $A \sin(\omega t) + B \cos(\omega t) = \sqrt{A^2 + B^2} \cos(\omega t + \phi)$, where $\phi$ is a phase angle. Similarly, $v_3$ and $v_4$ represent the second intrinsic mode (942.08 Hz).

Figure 7 shows the frequency spectrum of the mixed-mode impulse response. It can be seen that the decomposed oscillation frequencies (1187.84 Hz and 942.08 Hz) from SVD are relatively close to the dominant frequencies (1186.13 and 938.67 Hz) in the impulse response.
The decomposed modes after the first four components do not possess a consistent convergent dynamical process and are not shown. However, to evaluate their effects and determine how many modes are needed for accurate response modeling, we tested the results by using the first four and first six SVD components.

![Figure 6](image6.png)  
**Figure 6:** Plots of decomposed SVD components, $v_1$ to $v_4$.

One difficulty with learning oscillational responses is that to learn an ODE, each point on the trajectories must be unique over time. However, for trajectories that exhibit oscillation, the points along the trajectory are not unique, as shown in Figure 8, where multiple points on the curve at different times can take exactly the same value.

![Figure 7](image7.png)  
**Figure 7:** The frequency spectrum of the impulse response.

To uniquely define an oscillational trajectory, data with more than one dimension are needed. In order to solve this issue, a NODE model with two-dimensional (2D) inputs is used for the ODE learning. Since each pair of the temporal modes in $V$ correspond essentially to one intrinsic mode, a pair of temporal components can be learned together by a two-dimensional NODE model. Thus, a two-dimensional ODE can be learned with the combination of SVD components $v_1$ and $v_2$ for the first mode, components $v_3$ and $v_4$ for the second mode, etc. This allows, in most scenarios, a unique state along the entire trajectory; the trajectory does not cross itself at separate times. Figure 9 shows the two-dimensional trajectory of SVD components $v_1$ and $v_2$, with the red asterisk showing the initial condition at $t = 0$.

![Figure 8](image8.png)  
**Figure 8:** SVD component 1, $v_1$.

![Figure 9](image9.png)  
**Figure 9:** 2D system of SVD components, $v_1$ and $v_2$.

A two-dimensional NODE simply uses two-dimensional vectors as the input and output:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = nn\left(\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}\right)
$$

(6)

$$
\begin{bmatrix}
x_{1,i}^{t_1} \\
x_{2,i}^{t_1}
\end{bmatrix} = ODESolver\left(nn\left(\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}\right), \begin{bmatrix}
x_1^{t_0} \\
x_2^{t_0}
\end{bmatrix}, t_0, t_1\right)
$$

(7)

The two-dimensional NODE obtains a unique solution in the functional space such that the system response is sufficiently learned. The learning process of the model is explained as follows. First, the model iteratively solves an initial value problem, where a random point along the curve is taken as the initial condition and then the model predicts a short segment forward in time. Then, this process is repeated for $m$ initial conditions in a batch. The loss is then calculated between the predicted response and the true response. This process is repeated until the loss converges. Finally, the
learned modes are reconstructed back to the original mixed-mode impulse response, as illustrated in Figure 3.

4. RESULTS AND DISCUSSIONS

This section presents the results of the proposed method for impulse response modeling. Section 4.1 presents the reconstruction results with NODEs to determine if the multi-modal response can be approximated from learned SVD components using full-length data. Then, in Section 4.2, we show the prediction results from the learned data-driven ODE function when only a small segment of data is used for training.

4.1. Case Study A: Learning of Multi-Modal Dynamics from SVD Components using Full-Length Data

The task of the first case study is to determine whether or not a multi-modal impulse response can be effectively reconstructed from learned SVD components. The measured impulse response of the end mill in the machine tool spindle, as described in Section 3.1, was used for this case study. For the first evaluation, the entire measured impulse response was used to perform the time-delay embedding and thereafter, the SVD. A total of 6000 data points, about 0.12 seconds of data, were used to construct a 1000-step time-delayed embedding of size [5000*1000], following Eq (5). The first six columns of the V matrix, which represent the first six temporal modes, were taken and learned by the two-dimensional NODE model. Those SVD components can also be called ‘principal component trajectories’, as in (Dylewsky et al., 2022), or simply ‘modes’. After each principal component trajectory is learned by a NODE, the impulse response was reconstructed with the original U and \( \Sigma \) matrices. Figure 10 shows the learned first SVD component, \( v_1 \), versus the decomposed mode. The reconstruction shows high accuracy, with slight accumulated frequency mismatch over a relatively long period.

![Figure 10: Learned SVD component \( v_1 \) with NODE.](image)

The NODE model exhibits some difficulties when learning the SVD components during the training process. The learned ODE model can fail to converge sufficiently: the loss can stop decreasing and remain at a value that is relatively high, resulting in convergence failures. To remedy this issue, the model is simply reinitialized and trained again. The primary reason is the optimization process can get stuck in a local minimum and fails to converge to a lower loss, due to poor initialization. Similar instability issues have also been reported by other ODE papers (Norcliffe et al., 2020).

Two different reconstructions were made with four or six respective SVD components. The product of the original \( U \), \( \Sigma \), and the new \( V_{pred} \) matrices are used to obtain the reconstructed results. Firstly, the four-component reconstruction was performed. The results are shown in Figure 11 and Figure 12. It can be seen that with four components, the reconstructed impulse response is mostly accurate in both the amplitude and frequency. The learned response has a slightly lower amplitude as can be seen when time trends to 0.04 s (see Figure 11). As the result is evaluated further in time for 0.1 s, the learned response follows an accurate decay trend as the true response (see Figure 12).

![Figure 11: Mixed-mode impulse response reconstruction on four SVD components over 0.04 s.](image)

![Figure 12: Mixed-mode impulse response reconstruction on four SVD components over 0.1 s.](image)

The reconstructed impulse responses with the first six SVD components are shown in Figure 13 (0.04 s window) and Figure 14 (0.1 s window). It can be seen in Figure 13 that the model again effectively learns the multi-modal response.
since the reconstructed result matches the true response quite well. As the time approaches 0.04 s (see Figure 13), it can be seen that there is a slight discrepancy in terms of amplitude. When the plot is extended to 0.1 s (see Figure 14), an overfitting trend develops as time progresses, as seen in the inset of Figure 14. In that case, the true response begins to dampen out while the learned response still exhibits strong multi-modal characteristics.

4.2. Case Study B: Prediction of the Multi-Modal Impulse Response using Less Data

After proving that the multi-modal response could be learned by the proposed learning methods, the remaining question is whether this model is capable of predicting the response into the future. In order to evaluate this, a time-delay embedding was performed over only the first 1500 points instead of the full 6000 points. This translated to about 0.03 s of data. In this case, 500 delay steps are used, resulting in the columns of $V$ with a length of only 1000 points (about 0.02 s of data).

As previously discussed, the components beyond the first four principal component trajectories led to model inaccuracies as time progresses. Therefore, only the first four SVD components were used for this case study.

The components were learned in the same manner as before, except that the NODE model was trained over only about 0.02 s of data, which equates to 1000 points along the $V$ matrix. Next, the model performed predictions up to about 0.1 s, or 5000 points in length. When the reconstruction was performed, the original $\Lambda$ matrices had to be padded with zeros to predict up to a time of 0.1 s.

The reconstructed prediction from the first four principal component trajectories is shown in Figure 15 and Figure 16. Figure 15 shows that in the near term (0.04 s), the prediction captures fairly well the amplitude and the frequency of the multi-modal response. Figure 16 shows the long-term predicted response versus the true response. It can be seen that the predicted response matches the true response with relatively minor differences in amplitude after 0.04 s. In contrast, it can also be seen that the predicted response has a slightly larger amplitude compared to the true impulse response as time progresses.
Additionally, it was determined that prediction generally decreases for longer prediction times. Also, the accuracy improves when a longer segment is provided for training and learning.

In the future, more robust mode separation methods, for example, to learn the implicit Laplacian basis, can be investigated besides SVD to get clean separated mode. Furthermore, continuous in-situ response predictions with external excitations can be approached by uniting the proposed methods within a convolutional kernel.

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