

Uncertainty Quantification in Fatigue Damage Prognosis

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ABSTRACT

This paper presents a methodology to quantify the uncertainty in fatigue damage prognosis, applied to structures with complicated geometry and subjected to variable amplitude multi-axial loading. Finite element analysis is used to address the complicated geometry and calculate the stress intensity factors. Multi-modal stress intensity factors due to multi-axial loading are combined to calculate an equivalent stress intensity factor using a characteristic plane approach. Crack growth under variable amplitude loading is modeled using a modified Paris law that includes Wheeler's crack retardation model. During cycle-by-cycle integration of the crack growth law, a Gaussian process surrogate model is used to replace the expensive finite element analysis, resulting in rapid computation. The effect of different kinds of uncertainty – physical variability, data uncertainty and modeling errors – on crack growth prediction is investigated. The various sources of uncertainty include, but not limited to, variability in loading conditions, material parameters, experimental data, model uncertainty, etc. Three different kinds of modeling errors – crack growth model error, discretization error and surrogate model error – are included in analysis. The different kinds of uncertainty are incorporated into the prognosis methodology to predict the probability distribution of crack size as a function of number of load cycles. The proposed method is illustrated using an application problem, surface cracking in a cylindrical structure.

1. INTRODUCTION

Mechanical components in engineering systems are often subjected to cyclic loads leading to fatigue, crack initiation and progressive crack growth. It is essential to predict the performance of such components to facilitate risk assessment and management, inspection and maintenance

scheduling and operational decision-making. Researchers have pursued two different kinds of methodologies for fatigue life prediction. The first method is based on material testing (to generate S-N, ϵ -N curves) and use of an assumed damage accumulation rule. In this method, specimens are subjected to repeated cyclic loads under laboratory conditions. Hence the results are specific to the geometry of the structure as well as the nature of loading. Further, the performance of these components under field conditions is significantly different from laboratory observation, due to various sources of uncertainty accumulating in the field that render experimental studies less useful. Hence, this methodology cannot be used directly to predict the fatigue life of practical applications wherein complicated structures subjected to multi-axial loading.

The second method for fatigue life prediction is based on principles of fracture mechanics and crack growth analysis. A crack growth law is assumed and the progressive growth of the crack is modeled. However, this is not straightforward. Fatigue crack growth is a stochastic process and there are different kinds of uncertainty – physical variability, data uncertainty and modeling errors, associated with it. Uncertainty appears at different stages of analysis and the interaction between these sources of uncertainty cannot be modeled easily. Further, the application of crack growth principles to complicated structures, subjected to multi-axial variable amplitude loading requires repeated evaluation of finite element analysis which makes the computation expensive.

Some of these problems have been investigated by researchers in detail. The first problem in using a crack growth model is that the initial crack size is not known. This issue is further complicated by the fact that small crack growth propagation is anomalous in nature (Liu and Mahadevan, 2008). This problem was addressed by the introduction of an equivalent initial flaw size (EIFS) nearly thirty years ago. The concept of EIFS was introduced to by-pass small crack growth analysis and to substitute an initial crack size in long crack growth

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models such as Paris' law. However EIFS does not represent any physical quantity and cannot be measured using experiments. Initially, certain researchers used empirical crack lengths between 0.25 mm and 1 mm for metals (JSSG, 2006; Gallagher et al., 1984; Merati et al., 2007). Later, several researchers (Yang, 1980; Moreira et al., 2000; Fawaz, 2000; White et al., 2005; Molent et al., 2006) used back-extrapolation techniques to estimate the value for equivalent initial flaw size. Recently, Liu and Mahadevan (Liu and Mahadevan, 2008) proposed a methodology based on the Kitagawa-Takahashi diagram (Kitagawa and Takahashi, 1976) and the El-Haddad Model (Haddad et al., 1979) to derive an analytical expression for the equivalent initial flaw size. The current research work uses this concept to calculate the statistics of EIFS from material properties such as threshold stress intensity factor and fatigue limit. These material properties are calculated from experimental data and the associated data uncertainty due to measurement errors, sparseness of data, etc. needs to be taken into account.

The next step in damage prognosis is to choose a crack growth model. There are many crack growth models available in literature. In this paper, a modified Paris law is used as the crack growth law for the sake of illustration, but an error term (treated as a random variable) is added to represent the fitting error since experimental data were used to estimate the coefficients of the Paris model. Further, the model coefficients are also treated as random variables. The effects of variable amplitude loading are considered by using a Wheeler's retardation model in conjunction with modified Paris' law.

The modified Paris law based on linear elastic fracture mechanics calculates the increase in crack size as a function of the stress intensity factor, during each loading cycle. The stress intensity factor, in turn, is a function of the current crack size, crack configuration, geometry of the structural component and loading conditions. If structures with complicated geometry are subjected to multi-axial loading, then the stress intensity factor needs to be calculated through expensive finite element analysis, at every loading cycle. This paper replaces the finite element analysis with a surrogate model, known as the Gaussian process (GP) interpolation. Several finite element analysis runs are used to train this surrogate model and then, the surrogate model is used to predict the stress intensity factor, to be used in the crack growth law. There are two kinds of errors in this procedure. First, the finite element analysis has discretization error that needs to be accounted for while training the surrogate model. Second, the surrogate model adds further uncertainty since it is obtained by fitting the model to the (finite element) training data.

In addition to the above mentioned model uncertainty and data uncertainty (used to calculate the EIFS), natural variability in many input variables introduces uncertainty in model output. The loading on the structure is usually random in nature. A variable amplitude multi-axial loading history consisting of bending and torsion is illustrated in this paper.

Natural variability also includes variability in material properties, geometry and boundary conditions. The variability in certain material properties such as fatigue limit and threshold stress intensity factor is considered while deriving the statistical distribution of EIFS. The

geometry of the specimen and boundary conditions are considered deterministic in this research work.

The main focus of this paper is to investigate in detail each source of uncertainty and propose a methodology that can effectively account for all of them. Finally, the developed framework is used to predict the probabilistic fatigue life of the structure.

The next section reviews the existing literature on this topic. Section 3 presents the algorithm used in this paper to predict the fatigue life of structures with complicated geometry and subjected to variable amplitude, multi-axial loading. The various sources of uncertainty in this procedure are discussed in Section 4. Section 5 presents the proposed framework for uncertainty quantification in crack growth prediction. Section 6 illustrates the methodology through an example, considering cracking in a cylindrical structure.

2. LITERATURE REVIEW

Numerous studies have addressed probabilistic crack growth and life prediction, but focused mainly on natural variability in loading, geometry and material properties. The "damage prognosis" project at Los Alamos national laboratory (Doebling and Hemez, 2001; Hemez et al., 2003; Farrar et al., 2004; Farrar and Lieven, 2006) has addressed this problem in detail and researchers have proposed probabilistic methods as a solution to this problem. Sampling techniques were used to predict crack growth in composite plates and the error between prediction and observation was also characterized. Loading (uniaxial impact loading) conditions and geometric and material properties are treated as random variables. Surrogate models were used to replace expensive finite element models, and included in a sampling based framework for uncertainty propagation. Finite element analysis results were used to train the surrogate models, but the discretization error was not quantified. Further, the errors due to usage of surrogate models, errors in crack growth model, etc. were not addressed.

Besterfield et al. (Besterfield et al., 1991) combined probabilistic finite element analysis with reliability analysis to predict crack growth in plates. Random mixed mode loading cycles, physical variability in material properties, randomness in crack configuration (size, position and angle) were considered. However, the implementation of probabilistic finite element analysis is computationally expensive for structures with complicated geometry. Other sources of uncertainty such as data uncertainty and model uncertainty were not considered.

Patrick et al (Patrick et al., 2007) introduced an online fault diagnosis and failure prognosis methodology applied to a helicopter transmission component. A crack growth model (Paris law) was used for fatigue life prediction. Bayesian techniques were implemented to infer the initial crack size, which was used for probabilistic fatigue life prediction using particle filter techniques. Other sources of uncertainty such as error in Paris law,

variability in model parameters, and randomness in loading were not considered.

Gupta and Ray (Gupta and Ray, 2007) developed algorithms for online fatigue life estimation that relied on time series data analysis of ultrasonic signals and were built on the principles of symbolic dynamics, information theory and statistical pattern recognition. Physical variability in material geometry (surface defects, voids, inclusions, sub-surface defects), minor fluctuations in environmental conditions and operating conditions were used to quantify the uncertainty in detection which was further used to quantify the uncertainty in prognosis.

Pierce et al (Pierce et al., 2007) discussed the application of interval set techniques to the quantification of uncertainty in a neural network regression model of fatigue life, applied to glass fiber composite sandwich materials. This paper only considered the uncertainty in input data and other sources of uncertainty were not investigated in detail.

Orchard et al (Orchard et al., 2008) used the method of particle filters for uncertainty management in fatigue prediction. However, the various sources of uncertainty were not clearly delineated and considered in the analysis. While the use of conditional probability has been recommended for probabilistic predictions, this turns out to be expensive when variable amplitude loading cycles are considered, as the ensemble of predictions grows in size as a function of the number of loading cycles.

Papazian et al (Papazian et al., 2009) developed a structural integrity prognosis system (SIPS), based on collaboration between sensor systems and advanced reasoning methods for data fusion and signal interpretation, and modeling and simulation. Probabilistic principles such as likelihood and conditional probability were used to compare model predictions and sensor data. While measurement errors and sensor data were considered in detail, solution errors, variability of model parameters, randomness in loading, etc were not considered.

Thus past studies on uncertainty quantification in prognosis problems have ignored several sources of uncertainty or not investigated them in detail. Physical variability (such as randomness in loading conditions, material parameters, etc.) has been mainly studied by researchers, whereas other sources of uncertainty such as data uncertainty and model uncertainty have not been fully addressed.

This paper proposes a framework which can effectively account for different sources of uncertainty – physical variability, data uncertainty and model uncertainty. The various sources of uncertainty are discussed in detail, later in Section 4. Prior to that, the algorithm for crack growth propagation is outlined in the following section.

3. CRACK GROWTH PROPAGATION

Consider the growth of an elliptic crack. A schematic of the crack growth is shown in Fig. 1.

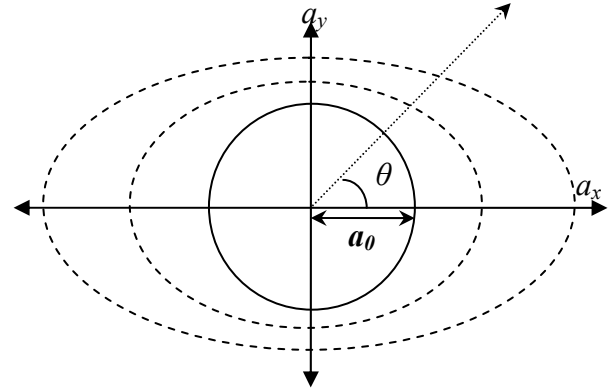


Fig. 1. Elliptic Crack Growth

In Fig. 1, a_x denotes the length of the semi-major axis and a_y denotes the length of the semi-minor axis. The aspect ratio, calculated as ratio between a_x and a_y is denoted by γ . If θ denote the angle of orientation, then a_x corresponds to $\theta = 0^\circ$ and a_y corresponds to $\theta = 90^\circ$. Crack growth laws such as Paris law (applicable to long cracks) predict the increase in crack size as a function of stress intensity factor, which in turn depends on the current crack size (a_x, a_y), aspect ratio (γ), angle of orientation (θ) and loading (L). In this paper, a has been used to denote the crack size in two directions, i.e. $\mathbf{a} = [a_x, a_y]$. Hence, the two dimensional array \mathbf{a} contains information about aspect ratio (β) as well.

Starting with an initial crack size (\mathbf{a}_0), the growth of the crack can be modeled and the crack size after a given number of cycles can be calculated. However, the initial crack size cannot be calculated exactly. The concept of EIFS was proposed to tackle this problem. Starting with the introduction of EIFS, this section explains the various steps involved in using a crack growth model to predict the crack size as a function of number of cycles.

3.1 Use of EIFS in Crack Growth Law

The rigorous approach to fatigue life prediction would be to perform crack growth analysis starting from the actual initial flaw, accounting for voids and non-metallic inclusions. If the initial crack size is large, then long crack growth models such as Paris' law can be used directly. However, this is not the case in most materials. Hence the long crack growth model cannot be used directly.

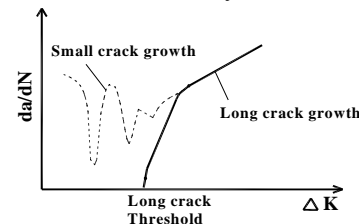


Fig. 2. Schematic of Crack Growth

A schematic plot of the long crack and short crack growth curves is given in Fig. 2.

Consider any long crack growth law used to describe the relationship between da/dN and ΔK , where N represents the number of cycles, a represents the crack size and ΔK represents the stress intensity factor. This paper uses modified Paris' law for illustration purposes and includes the effects of Wheeler's retardation model as:

$$da/dN = \phi^r C (\Delta K)^n (1 - \Delta K_{th}/\Delta K)^m \quad (1)$$

In Eq. (1), ϕ^r refers to the retardation parameter (Sheu et al., 1995), and is equal to unity if $a_i + r_{p,i} > a_{OL} + r_{p,OL}$ where a_{OL} is the crack length at which the overload is applied, a_i is the current crack length, $r_{p,OL}$ is the size of the plastic zone produced by the overload at a_{OL} , and $r_{p,i}$ is the size of the plastic zone produced at the current crack length a_i . Else, ϕ^r is calculated as shown in Eq. (2).

$$\phi^r = (r_{p,i} / (a_{OL} + r_{p,OL} - a_i))^\lambda \quad (2)$$

In Eq. (2), λ is the curve fitting parameter for the original Wheeler model termed the shaping exponent [Yuen et al., 2006]. Sheu et al. [Sheu et al., 1995] and Song et al. [Song et al., 2001] observed that crack growth retardation actually takes place within an effective plastic zone. Hence the size of the plastic zone can be calculated in terms of the applied stress intensity factor (K) and yield strength (σ) as:

$$r_p = \alpha (K/\sigma)^2 \quad (3)$$

In Eq. (3), α is known as the effective plastic zone size constant which is calculated experimentally (Yuen et al., 2006). The expressions in Eq. (2) and Eq. (3) can be combined with Eq. (1) and used to calculate the crack growth as a function of number of cycles. In each cycle, the stress intensity factor can be expressed as a function of the crack size (a), loading (L) and angle of orientation (θ). Hence, the crack growth law in Eq. (1) can be rewritten as

$$da/dN = g(a, L, \theta) \quad (4)$$

The concept of an equivalent initial flaw size was proposed to bypass small crack growth analysis and make direct use of a long crack growth law for fatigue life prediction. The equivalent initial flaw size, a_0 is calculated from material properties (ΔK_{th} , the threshold stress intensity factor and σ_f , the fatigue limit) and geometric properties (Y) as explained in Liu and Mahadevan (Liu and Mahadevan, 2008).

$$a_0 = (1/\pi)(\Delta K_{th} / Y\sigma_f)^2 \quad (5)$$

By integrating the expression in Eq. (1), the number of cycles (N) to reach a particular crack size a_N can be calculated as shown in Eq. (6).

$$N = \int da / g(a, L, \theta) = \int da / (\phi^r C (\Delta K)^n (1 - \Delta K_{th}/\Delta K)^m) \quad (6)$$

For structures with complicated geometry and loading conditions, the integral in Eq. (6) is to be evaluated cycle by cycle, calculating the stress intensity factor in each cycle of the crack growth analysis. The calculation of the

stress intensity factor is explained in the following subsection.

3.2 Calculation of Stress Intensity Factor

The stress intensity factor ΔK in Eq. (6) can be expressed as a closed form function of the crack size for specimens with simple geometry subjected to constant amplitude loading. However, this is not the case in many mechanical components, where ΔK depends on the loading conditions, geometry and the crack size. Further, if the loading is multi-axial (for example, simultaneous tension, torsion and bending), then the stress intensity factors corresponding to three modes need to be taken into account. This can be accomplished using an equivalent stress intensity factor. If K_I , K_{II} , K_{III} represent the mode-I, mode-II and mode-III stress intensity factors respectively, then the equivalent stress intensity factor K_{eqv} can be calculated using a characteristic plane approach proposed by Liu and Mahadevan (Liu and Mahadevan, 2005). The use of the characteristic plane approach for crack growth prediction under multi-axial variable amplitude loading has been validated earlier with several data sets.

During each cycle of loading, the crack grows and hence, the stress intensity factor needs to be reevaluated at the new crack size for the loading in the next cycle. Hence, it becomes necessary to integrate the expression in Eq. (6) through a cycle by cycle procedure. Each cycle involves the computation of ΔK using a finite element analysis represented by Ψ .

$$\Delta K_{eqv} = \Psi(a, L, \theta) \quad (7)$$

Repeated evaluation of the finite element analysis in Eq. (7) renders the aforementioned cycle by cycle integration extremely expensive, perhaps impossible in some cases. Hence, it is necessary to substitute the finite element evaluation by an inexpensive surrogate model. Different kinds of surrogate models (polynomial chaos, support vector regression, relevance vector regression, and Gaussian Process interpolation) have been explored and the Gaussian process modeling technique has been employed in this paper. A few runs of the finite element analysis are used to train this surrogate model and then, this model is used to predict the stress intensity factor for other crack sizes and loading cases (for which finite element analysis has not been carried out).

3.3 Gaussian Process Surrogate Modeling

A Gaussian process (GP) response surface approximation is constructed to capture the relationship between the input variables (a , L , θ) and the output variables (ΔK) in Eq. (5), using only a few sample points within the design space. The details of this interpolation technique are available in literature (Rasmussen, 1996; Santner, 2003; McFarland, 2007).

The basic idea of the GP model is that the response values Y (K_{eqv} in this case), are modeled as a group of multivariate normal random variables, with a defined mean and covariance function. The benefits of GP modeling is that the method requires only a small number of sample points (usually 30 or less), and is capable of capturing highly nonlinear relationships that exist between input and output variables without the need for an explicit functional form. Additionally, Gaussian process models can be used to fit virtually any functional form and provide a direct estimate of the uncertainty associated with all predictions in terms of model variance. The framework of Gaussian process modeling is shown in Fig. 3.

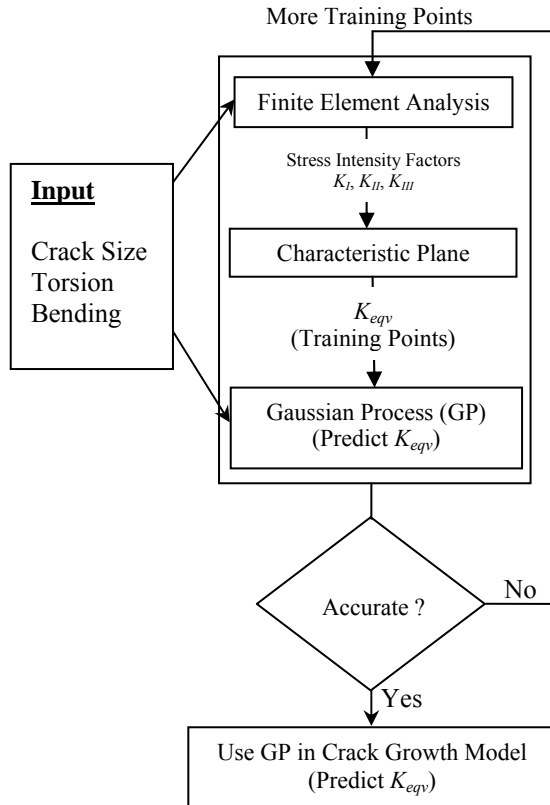


Fig. 3. Construction of Surrogate Model

Suppose that there are m training points, $x_1, x_2, x_3 \dots x_n$ of a d -dimensional input variable (the input variables being the crack size and loading conditions here), yielding the resultant observed random vector $Y(x_1), Y(x_2), Y(x_3) \dots Y(x_n)$. R is the $m \times m$ matrix of correlations among the training points. Under the assumption that the parameters governing both the trend function ($f^T(x_i)$ at each training point) and the covariance (λ) are known, the conditional expected value of the process at an untested location x^* is calculated as in Eq. (8) and Eq. (9) respectively.

$$Y^* = E(Y|x^*) = f^T(x^*)\beta + r^T(x^*)R^{-1}(Y - F\beta) \quad (8)$$

$$\sigma_{Y^*} = \text{Var}(Y|x^*) = \lambda(1 - r^T R^{-1} r) \quad (9)$$

In Eq. (8) and Eq. (9), F is a matrix with rows $f^T(x_i)$, r is the vector of correlations between x^* and each of the

training points, β represents the coefficients of the regression trend. McFarland (McFarland, 2007) discusses the implementation of this method in complete detail.

3.4 Crack Propagation Analysis

This section explains the method used to calculate the final crack size as a function of number of load cycles. The procedure involves the evaluation of the integral in Eq. (4). As explained in Section 3.3, this needs to be done cycle by cycle and the Gaussian process surrogate model is used to predict the equivalent stress intensity factor in each cycle.

Starting with the equivalent initial flaw size a_0 , the equations (Eq. (1) – Eq. (6)) described in Section 3.1 are used to calculate the final crack size A after N loading cycles. This entire procedure is summarized in Fig. 4.

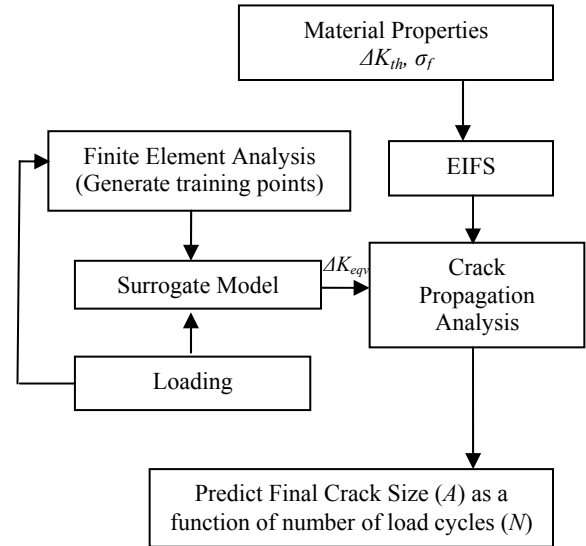


Fig. 4. Crack Propagation Analysis

The framework shown in Fig. 4 for prognosis is deterministic and does not account for errors and uncertainty. Uncertainty can be associated with each of the blocks in Fig. 4 and accounted for in prognosis. The following section investigates these sources of uncertainty and Section 5 incorporates them into the prognosis framework.

4. SOURCES OF UNCERTAINTY

This section discusses the various sources of uncertainty and errors that are part of the prognosis framework summarized in Section 3.5 and proposes methods to handle different kinds of uncertainty. The material properties used to calculate the equivalent initial flaw size are measured using experiments and have variability, causing variability in EIFS. Further, these experimental data may be sparse and the uncertainty in data needs to be accounted for. The crack growth law used for crack propagation is usually estimated through curve

fitting of experimental data. To account for model uncertainty, a (normally distributed) error term is added to the crack growth equation and the model coefficients of the crack growth law are treated as random variables. In each cycle of loading, the stress intensity factor is calculated as a function of current crack size, loading and geometry. Repeated finite element analyses are avoided by the use of inexpensive surrogate models and the output of the surrogate model is not accurate. Further, the training points calculated using finite element analyses are prone to solution approximation and discretization errors. Further, the loading itself is considered to be random – a variable amplitude multi-axial loading case is demonstrated in this paper. These various sources of uncertainty can be classified into three different types – physical variability, data uncertainty and model uncertainty - as shown below.

- I. Physical Variability
 - a. Loading
 - b. Equivalent initial flaw size
 - c. Material Properties (Fatigue Limit, Threshold Stress Intensity Factor)
- II. Data Uncertainty
 - a. Material Properties (Fatigue Limit, Threshold Stress Intensity Factor)
- III. Model Uncertainty/Errors
 - a. Crack growth law uncertainty
 - b. Uncertainty in calculation of Stress Intensity factor
 - A. Discretization error in finite element analysis
 - B. Uncertainty in surrogate model output

(Note: Variations in geometry and boundary conditions are sources of physical variability. These variations are not considered in this research work. However, these can be included in the proposed framework by constructing different finite element models (for different geometry and boundary conditions) and use these runs to train the Gaussian process surrogate model. Hence, these parameters are treated as inputs to the surrogate model and sampled randomly in the uncertainty quantification procedure explained later in Section 5.)

The following subsections discuss each source of uncertainty in detail and propose methods to handle them.

4.1 Physical Variability in Loading Conditions

The loading on practical structures is rarely deterministic and it is difficult to quantify the uncertainty in loading. For the purpose of illustration, variable amplitude multi-axial (bending, tension and torsion) loading is considered in this paper.

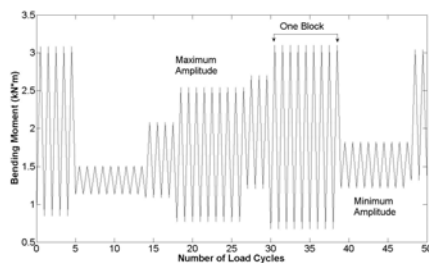


Fig. 5. Sample loading history

A loading history consists of a series of blocks of loads, the loading amplitude being constant in each block. In this paper, the block length is assumed to be a random variable and the maximum and minimum amplitudes in each block are also treated as random variables. A sample loading history is shown in Fig. 5.

To generate one block of loading, first a block length is selected and then a maximum amplitude value and a minimum amplitude value is selected for that block. The entire loading history is generated by repeating this process and creating several successive blocks.

4.2 Physical Variability in EIFS

The equivalent initial flaw size derived in Eq. (3) depends on ΔK_{th} , the equivalent mode-I threshold stress intensity factor, $\Delta\sigma_f$, the fatigue limit of the specimen and the geometry factor Y which in turn depends on the geometry of the structural component and the configuration of the crack. This is a deterministic quantity and can be estimated using finite element analysis. The distributions for the material properties, ΔK_{th} and $\Delta\sigma_f$ are characterized using data obtained from experimental testing. This is explained in Section 4.3. Having obtained the statistical distributions of ΔK_{th} and $\Delta\sigma_f$, the distribution of a_0 , the equivalent initial flaw size, can be calculated.

4.3 Data Uncertainty in Material Properties (to characterize distributions ΔK_{th} and $\Delta\sigma_f$)

This section proposes a general methodology to characterize uncertainty in input data, from which statistical distributions need to be inferred. This method is illustrated using experimental data available in literature to characterize the distribution of threshold stress intensity factor (ΔK_{th}) and fatigue limit ($\Delta\sigma_f$). McDonald et al. (McDonald et al., 2009) proposed a method to account for data uncertainty, in which in the quantity of interest can be represented using a probability distribution, whose parameters are in turn represented by probability distributions.

Consider a random variable X whose statistics are to be determined from experimental data, given by $\mathbf{x} = \{x_1, x_2 \dots x_n\}$. For the sake of illustration, suppose that the random variable X follows a normal distribution, then the parameters (\mathbf{P}) of this distribution, i.e. mean and variance of X can be estimated from the entire data set \mathbf{x} . However, due to sparseness of data, these estimates of mean and variance are not accurate. Using resampling techniques such as bootstrapping method, jackknifing etc. the probability distributions ($f_p(\mathbf{P})$) of the parameters (\mathbf{P}) can be calculated. Hence for each instance of a set of parameters (\mathbf{P}), X is defined by a particular normal distribution. However, because the parameters (\mathbf{P}) themselves are stochastic, X is defined by a family of normal distributions. For a detailed implementation of this methodology, refer McDonald et al., 2009.

This paper uses similar resampling techniques to calculate the distribution of the parameters (\mathbf{P}), however does not define a family of distributions. Instead, it recalculates the distribution of the random variable X , using principles of conditional probability (Haldar and Mahadevan, 2000).

Thus X follows a probability distribution conditioned on the set of parameters (\mathbf{P}). Hence the distribution of X is denoted by $f_{X|\mathbf{P}}(x)$. However, in this case, the parameters are represented by probability distributions $f_{\mathbf{P}}(\mathbf{P})$. Hence, the unconditional probability distribution of X ($f_X(x)$) can be calculated as shown in Eq. (10).

$$f_X(x) = \int f_{X|\mathbf{P}}(x) f_{\mathbf{P}}(\mathbf{P}) d\mathbf{P} \quad (10)$$

The integral in Eq. (10) can be evaluated through quadrature techniques or advanced sampling methods such as Monte Carlo integration or Markov chain Monte Carlo Integration. Hence, the unconditional distribution of X which accounts for uncertainty in input data can be calculated. In this paper, this method has been used to characterize the uncertainty in threshold stress intensity factor (ΔK_{th}) and fatigue limit ($\Delta\sigma_f$).

4.4 Uncertainty in Crack Growth Model

There are more than 20 different crack growth laws (e.g., Paris law, Foreman's equation, Weertman's equation) proposed in literature. The mere presence of many such different models explains that none of these models can be applied universally to all fatigue crack growth problems. Each of these models has its own limitations and uncertainty. In this paper, a modified Paris law has been used for illustration, however, the methodology can be implemented using any kind of crack growth model.

The uncertainty in crack growth model can be subdivided into two different types: crack growth model error and uncertainty in model coefficients. If ε_{cg} is used to denote the crack growth model error, then Paris law can be expressed as in Eq. (11).

$$da/dN = c(\Delta K)^m + \varepsilon_{cg} \quad (11)$$

An estimate of ε_{cg} can be obtained while calibrating the model parameters using statistical data fitting tools. The model coefficients in Paris law are C and n , and the uncertainty in these parameters can be represented through probability distributions. The stress intensity factor ΔK , as explained earlier is calculated using the Gaussian process surrogate model as explained in Section 3. The various sources of uncertainty in this process are addressed in Section 4.5.

4.5 Errors in Stress Intensity Factor Calculation

As explained in Section 3, a Gaussian process model is used to calculate the stress intensity factor ΔK . This is done in two stages. First, a few finite element analysis runs are required to train the GP model. Second, the GP model is used to predict the stress intensity factor as explained in Section 3.3. Each of these two steps has associated errors and uncertainty. Finite element solutions are subject to discretization errors, whereas the prediction of any low-fidelity model such as the GP model also has error. These two issues are discussed in this subsection.

4.5.1 Discretization Error in Finite Element Analysis

Theoretically, an infinitesimally small mesh size will lead to the exact solutions but this is difficult to implement in practice. Hence, finite element analyses are carried at a particular mesh size and the error in the solution, caused due to discretization needs to be quantified. Several methods are available in literature but many of them quantify some surrogate measure of error to facilitate adaptive mesh refinement. The Richardson extrapolation (RE) method has been found to come closest to quantifying the actual discretization error and this method has been extended to stochastic finite element analysis by Rebba (Richards, 1997; Rebba, 2005). It should be noted that the use of Richardson extrapolation to calculate discretization error requires the model solution to be convergent and the domain to be discretized uniformly (uniform meshing) (Rebba et al., 2004). Sometimes, in the case of coarse models, the assumption of monotone truncation error convergence is not valid.

In the Richardson extrapolation method, the discretization error due to grid size, for a coarse mesh is given by Eq. (12).

$$\varepsilon_h = (f_1 - f_2)/(r^p - 1) \quad (12)$$

In Eq. (12), f_1 and f_2 are solutions for a coarse mesh and a fine mesh respectively. If the corresponding mesh sizes were denoted by h_1 and h_2 , then the grid refinement ratio, denoted by r is calculated as h_2/h_1 . The order of convergence of p is calculated as:

$$p = \log((f_3 - f_2)/(f_2 - f_1))/(\log(r)) \quad (13)$$

In Eq. (13), f_3 represents the solution for a coarse mesh of size h_3 , with the same grid refinement ratio, i.e. $r = h_3/h_2$.

The solutions f_1 , f_2 , f_3 are dependent on the inputs (loading, current crack size, aspect ratio and angle of orientation) to the finite element analysis and hence the error estimates are also functions of these input variables. For each set of inputs, a corresponding error is calculated and this error is added to the (coarse mesh) solution from finite element analysis to calculate the true solution. Hence a true solution is associated with each set of inputs and these values are used as training points for the surrogate model.

4.5.2 Uncertainty in the Surrogate Model Output

Several finite element runs for some combination of input-output variable values are used to train the Gaussian process surrogate model in this paper. Then, these surrogate models can be used to evaluate the stress intensity factor for other combinations of input variable values. GP models, as explained in Section 3.3, model the output as a sum of Gaussian variables and hence, inherently produce an output which is normally distributed. The expressions for mean and variance of the output

of the GP model were given in Eq. (8) and Eq. (9) respectively. The output of the GP (ΔK_{eq}) model is a random normal variable and in each cycle, the value for ΔK_{eq} is sampled from this distribution.

(Note: The GP model is used as a surrogate for the deterministic finite element model and the variance of the GP output accounts only for the uncertainty in replacing the original model with a Gaussian process and does not account for the uncertainty in the inputs to the model. The variance of the output is only dependant on the “form” of the surrogate model. For example, a linear surrogate model will lead to constant variance at untrained locations but unknown distribution type (Seber and Wild, 1989). The advantage is using a Gaussian process surrogate model is that not only the output variance can be calculated but also the distribution type can be proved to be Gaussian (McFarland, 23).)

The Gaussian process model output, i.e. the stress intensity factor is used in the crack growth equation to predict the crack size as a function of number of cycles as explained earlier in Section 3. The following section incorporates all these sources of uncertainty into the prognosis methodology described in Section 3.

5. UNCERTAINTY IN PROGNOSIS

Section 3 proposed a methodology that can be used for damage prognosis of structures with complicated geometry and subjected to multi-axial loading. This procedure was summarized using a step-by-step flowchart in Fig. 4. Section 4 investigated the various sources of uncertainty in the prognosis framework and proposed methods to handle them. A brief summary of the various sources of uncertainty is given below.

I. PHYSICAL VARIABILITY

- a. Variable amplitude multi-axial loading cycles are generated by considering random block lengths and random amplitudes within each block.
- b. The equivalent initial flaw size (EIFS) is represented by a probability distribution that accounts for the variability in material parameters, the threshold stress intensity factor and fatigue limit.
- c. The material properties (fatigue limit, threshold stress intensity factor) are represented by probability distributions, inferred from experimental data.

II. DATA UNCERTAINTY

- a. The uncertainty in data used to calculate the statistics of material properties (fatigue limit, threshold stress intensity factor) is addressed by using a sampling based approach that calculates a family of probability distributions for each material parameter. Then, this family of distributions is integrated into one single probability distribution (for each property) using the principles of conditional and total probability.

III. MODEL UNCERTAINTY/ERRORS

- a. The uncertainty in crack growth model is handled by adding an error term to the crack growth law and by representing the model parameters as random variables.

- b. The calculation of stress intensity factor in each cycle of crack growth is facilitated using a Gaussian process surrogate model.

- A. The discretization error in finite element analysis is calculated using Richardson extrapolation and added to the results of FEA before training the surrogate model.
- B. The uncertainty (calculated as the variance) in the surrogate model output is modeled as a Gaussian variable calculation from regression results and hence, the prediction of the surrogate model, i.e. the Stress intensity factor is represented as a normal distribution.

This section presents a sampling based strategy to combine all the different sources of uncertainty and thereby quantify the uncertainty in damage prognosis, i.e. the distribution of the final crack size is calculated as a function of number of loading cycles (N). The various steps in this procedure are outlined here.

- I. Generate training points for the Gaussian process surrogate model. This is done through finite element analysis and then by calculating the discretization error in each of the runs. The discretization errors are added to the solutions of finite element analysis and used to train the Gaussian process surrogate model. Hereon, the GP model can be used to calculate the stress intensity factor as a function of crack size, loading, aspect ratio and angle of orientation.
- II. Generate a loading history. First, randomly select a block length and then randomly select a maximum amplitude value and a minimum amplitude value for that particular block. Repeat the process till the number of cycles (N) is reached.
- III. Sample an EIFS value from the statistical distribution calculated in Section 4.1 and Section 4.2.
- IV. Use the deterministic prognosis methodology to calculate the final crack size at the end of N cycles. However, in each loading cycle, the stress intensity factor calculated from the GP model is a random normal variable and hence generate a random sample of stress intensity factor in each cycle. Also, the crack growth model error (ε_{cg}) is sampled in every cycle.

In this algorithm, Step I is a deterministic step while Step II, Step III and Step IV are probabilistic. Using this algorithm, the crack size after N cycles can be calculated for a particular load history that was generated in Step II. Using Monte Carlo Sampling, Steps II, III and IV can be repeated again and again, each leading to a final crack size at the end of N cycles. This can be used to characterize the distribution of final crack size at the end of N cycles. By varying N , the distribution of final crack size can

be obtained as a function of the number of cycles (N).

This information can be used to calculate the reliability of the structural component as a function of number of load cycles. Suppose that the component is supposed to have failed if the crack size is greater than a critical crack size (A_c), then the probability of failure can be calculated as a function of load cycles.

6. NUMERICAL EXAMPLE

This section illustrates the proposed methodology to quantify the uncertainty in damage prognosis through a numerical example.

6.1 Description of the Problem

A two radius hollow cylinder with an elliptical crack in fillet radius region is considered for this purpose. This problem consists of modeling an initial semi-circular surface crack configuration and allowing the crack shape to develop over time into a semi-elliptical surface crack. This is shown in Fig. 6.

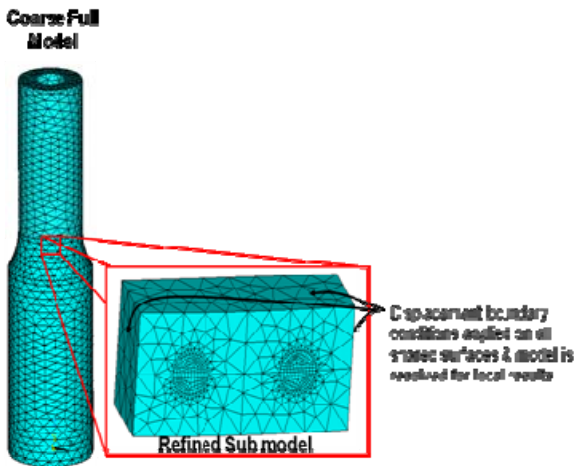


Fig. 6. Surface Crack in a Cylindrical Structure

The finite element software package ANSYS (ANSYS, 2007) version 11.0 is used to build and analyze the finite element model. The crack configuration is built by extruding a projection of the semi-circular crack through the main body at the crack location. The immediate volumes on either side of the crack face are identified and subdivided in order to allow for SIF evaluation at various locations along the crack front. The crack faces (coinciding upper and lower surfaces of the previously mentioned volumes) are then modeled as surface to surface contact elements (CONTACT174 and TARGET170 elements) in order to prevent the surface penetration of the crack's upper and lower surfaces. The augmented Lagrangian method is the algorithm used for contact simulation. Additionally, friction effect is included in the material properties of the contact element, in which a Coulomb friction model is used. This model defines an equivalent shear stress which is proportional to the contact pressure and the friction coefficient. Friction

coefficients between two crack faces are difficult to measure and are generally assumed to vary between 0 and 0.5 (Liu et al., 2007). The friction coefficient, μ , used within this study is assumed to be a deterministic quantity and taken to be equal to 0.1.

Since the primary quantity of interest is the stress intensity factor at the crack tip, the volume along the crack front is subdivided into many smaller blocks, which allows for better mesh control and enables SIF evaluation at various locations along the crack front. The mesh around the crack location (at the crack front and surrounding areas) is refined in order to obtain a more accurate solution and avoid convergence problems. To facilitate this, the crack region is constructed within a submodel of the uncracked body. The submodel technique is based on the St. Venant's principle, which states that if an actual distribution of forces is replaced by a statically equivalent system, the distribution of stress and strain is altered only near the regions of load application. It is observed that the result yields accurate stress intensity factor solutions all along the crack front which can be used for crack growth analysis.

Table 1 and Table 2 list the material and geometrical properties of the specimen under study.

Table 1 Material properties

Aluminium 7075- T6	
Modulus of Elasticity	72 GPa
Poisson Ratio	0.32
Yield Stress	450 MPa
Ultimate Stress	510 MPa

Table 2 Geometrical Properties

Cylinder Properties	
Length	152.4 mm
Inside Radius	8.76 mm
Outside Radius (Narrow Sect)	14.43 mm
Outside Radius (Wide Sect)	17.78 mm

In reality, these parameters in Table 2 and Table 3 may be variable and might require probabilistic treatment. However, as mentioned earlier, physical variability in the geometry of the structure, Young's modulus, Poisson ratio, boundary conditions, friction coefficient between crack faces, etc are treated to be deterministic in this paper.

The following subsection discusses the numerical implementation of the uncertainty quantification procedure.

6.2 Uncertainty in Prognosis

The numerical details of the different sources of uncertainty are presented in this section. They are given step-wise in the same order as in Section 5.

- I. Finite element analyses are run for 10 different crack sizes, 6 different loading cases, two angles of orientation and three different aspect ratios, amounting to 360 training points to construct the surrogate model. For each solution, three different meshes are considered and the discretization error is quantified as explained in Section 4.4.1. The discretization error is added to the finite element analysis solution at each training point and the Gaussian process model is trained to predict the stress intensity factor.
- II. Multi-axial variable amplitude loading cycles are generated by considering blocks of equal amplitude within one entire loading history. The block length is assumed to be a uniform distribution ($U(0,500)$) and the maximum amplitude and minimum amplitude for that block are assumed to follow normal distributions ($N(8,2)$ and $N(24,2)$) respectively, in KNm).
- III. The distribution of EIFS is characterized using the data used by Liu and Mahadevan (Liu and Mahadevan, 2008). However, the current research work accounts for uncertainty in data and treats the parameters of threshold stress intensity factor and fatigue limit as random variables as well. The distribution (conditioned on its parameters) of EIFS is assumed to be lognormal (with parameters λ , ζ), with the λ following a normal distribution (mean = -7.60 and standard deviation = 0.50) and ζ following a lognormal distribution (mean = 0.22mm and standard deviation = 0.10 mm). The unconditional distribution of EIFS is calculated using the integral in Eq. (8). Samples of EIFS are drawn from this distribution.
- IV. Paris law is used for crack growth propagation. The model parameter C (mean = 6.5×10^{-13} and standard deviation = 4×10^{-13}) is chosen to be lognormally distributed whereas m ($m = 3.9$) is treated as a deterministic quantity. These are identical to the distributions used by Liu and Mahadevan (Liu and Mahadevan, 2009). In each loading cycle, the values of stress intensity factor and crack growth model error (ϵ_{cg}) are sampled from probability distributions. While the stress intensity factor (calculated using the Gaussian process surrogate model) is a Gaussian variable (as explained in section 4.5.2), the crack growth model error is treated as a normal variable with zero mean and 0.05 coefficient of variation. The latter quantity is chosen to be normal (Seber and Wild, 1989) because it represents a fitting error while calculating the coefficients of modified Paris' law.

Using the sampling-based framework in Section 5, the probability distribution of the final crack size is calculated as a function of the total number of cycles. A Monte Carlo simulation using 5000 runs is used to calculate the probability distribution of crack size as a function of number of load cycles. The mean, median and 90%

prediction bounds of the final crack size are shown in Fig. 7.

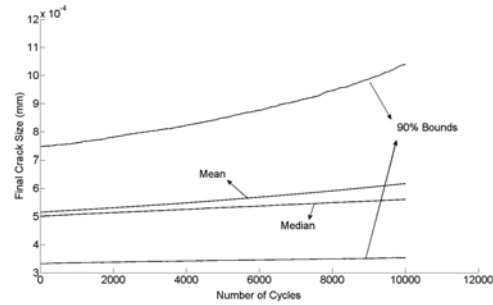


Fig. 7. Mean, Median and 90% Bounds

In Fig. 7, the growth of the crack is shown as a function of number of load cycles. As the number of cycles increase, there is more uncertainty and hence, the 90% prediction bounds are wider. This is due to the fact that each additional loading cycle imparts more randomness arising from variability in loading, variability in crack size at the end of previous cycle, uncertainty in the prediction of stress intensity factor, etc.

To illustrate the increase in uncertainty, the standard deviation of crack size is calculated as a function of number of load cycles and plotted in Fig. 8.

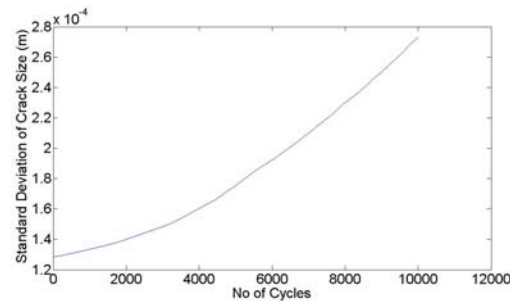


Fig. 8. Standard Deviation of Final Crack Size

Fig. 8 clearly shows the increase in uncertainty with number of load cycles. While the standard deviation of the initial crack size is low, it increases by about 500% at the end of 5000 load cycles. This increase is due to accumulation of different sources of uncertainty in each loading cycle, i.e. loading uncertainty, surrogate modeling errors and crack growth model errors.

Finally, the reliability of the structural component is also evaluated. A critical crack size of 2.54 mm (approximately 0.1 inch) is assumed and the probability of failure is estimated as a function of number of load cycles and plotted in Fig. 9. From Fig. 9, it is seen that the probability of failure is negligible for about 3500 load cycles and it gradually increases after 4000 cycles.

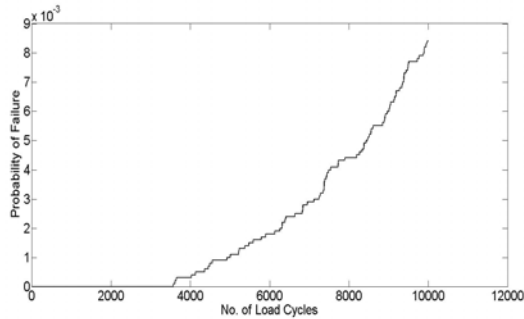


Fig. 9. Probability of Failure vs. No. Load Cycles.

There are two reasons for the observed increase in increase of failure probability. Firstly, the crack is growing in size and secondly, the uncertainty in the estimated crack size also increases with each loading cycle. After 10000 cycles of loading, the probability of failure is approximately equal to 0.01.

6.3 Individual Contributions of Uncertainty

The previous subsection presented the effect of all the different sources of uncertainty in the final distribution of crack size. The current subsection calculates the marginal contributions of each source of uncertainty in the overall results of prognosis. Such an analysis would identify which sources of uncertainty are critical and what the analyst must do in order to reduce the overall uncertainty in prognosis.

To calculate the contribution of one particular kind of uncertainty, all other quantities are assumed to be deterministic (at their mean values) and the results of this analysis are compared with the results of Section 6.2, where all sources of uncertainty were accounted. The individual contributions of each uncertainty are tabulated in Table 3.

Table 3. Individual Contributions of Uncertainty

Sources of Uncertainty Considered	Final Crack Size		
	Mean (mm)	Std (mm)	COV
All	0.0617	0.0273	0.4424
Loading	0.0592	0.0068	0.1152
Crack Growth Model	0.0544	0.0023	0.0421
Data Uncertainty	0.0547	0.0151	0.2767
EIFS Uncertainty	0.0544	0.0134	0.2463
GP Model Uncertainty	0.0544	5.33E-6	9.81E-6

It is seen that the contribution of uncertainty from loading is significant. This may be attribute to the fact the loading conditions in practical applications are random and variable to a great extent. Also, contribution due to errors in crack growth model is extremely small. This means that the variance in the parameters of Paris law used in this paper have little effect on the variance of final crack size. It is observed that the uncertainty due to sparse data is high. More data can be collected to reduce this kind of

uncertainty. The uncertainty due to the input equivalent initial flaw size (which includes uncertainty due to sparse data) is high). More experimental data can be collected and EIFS can be calibrated to reduce the uncertainty in the estimate of EIFS. This would also reduce the data uncertainty and decrease in the uncertainty in the estimate of final crack size in prognosis.

7. SUMMARY

This paper investigated the various sources of uncertainty in a fatigue damage prognosis problem and illustrated the proposed methods to quantify the overall uncertainty in crack growth prediction for structures with complicated geometry and multi-axial loading. The concept of equivalent initial flaw size was used to replace small crack growth analysis and use a long crack growth model, specifically Paris law, for crack propagation. Expensive finite element analysis was replaced by an inexpensive surrogate, i.e. the Gaussian process model, to evaluate the stress intensity factor in each cycle for use in crack growth law. Several sources of uncertainty – physical variability, data uncertainty and modeling errors - were included in prognosis. Physical variability included loading conditions and material properties such as threshold stress intensity factor and fatigue limit. The uncertainty in data used to characterize these parameters was accounted for. Three different kinds of modeling errors – discretization errors, surrogate modeling error and crack growth model error – were considered in this paper. A probabilistic methodology was proposed to incorporate these sources of uncertainty into the prognosis framework. A Monte Carlo based sampling approach is used to calculate the distribution of crack size as a function of number of loading cycles. By defining a suitable serviceability criterion (for example, crack size being greater than a critical value), the reliability of the structural component is calculated a function of number of loading cycles.

This research work also reported the individual contributions of various sources of uncertainty to the overall uncertainty in prognosis. This kind of study is popularly called as global sensitivity analysis and the method presented in this paper is a heuristic approach only. Rigorous methods for sensitivity analysis have been developed by several researchers around the world and future work would involve the application of these methods to prognosis problems.

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NOMENCLATURE

a_0	Equivalent Initial Flaw Size
a_x	Crack size along x-direction
a_y	Crack Size along y-direction
N	Number of Loading Cycles
A	Final Crack Size after N loading cycles
L	Description of Load
ΔK_{th}	Threshold Stress Intensity Factor
σ_f	Fatigue Limit
ΔK	Stress Intensity Factor
Y	Geometry Factor
ϵ_{cg}	Crack Growth Model Error

REFERENCES

- Joint service specification guide aircraft structures, JSSG-2006. United States of America: Department of Defense; 1998.
- J.P. Gallagher, A.P. Berens, and R.M. Engle Jr. (1984). USAF damage tolerant design handbook: guidelines for the analysis and design of damage tolerant aircraft structures. Final report. 1984.
- A. Merati, and G. Eastaugh. (2007). Determination of fatigue related discontinuity state of 7000 series of aerospace aluminum alloys. *Eng Failure Anal* 2007; 14(4):673–85.
- J.N. Yang. (1980). Distribution of equivalent initial flaw size. In: Proceedings of the annual reliability and maintainability symposium. San Francisco (CA): 1980.
- P. White, L. Molent, and S. Barter. (2005). Interpreting fatigue test results using a probabilistic fracture approach. *Int J Fatigue* 2005;27(7):752–67.
- L. Molent, Q. Sun, and A. Green. (2006). Characterisation of equivalent initial flaw sizes in 7050 aluminium alloy. *J Fatigue Fract Eng Mater Struct* 2006;29:916–37.
- PMGP. Moreira, PFP. de Matos, and PMST. de Castro. (2005). Fatigue striation spacing and equivalent initial flaw size in Al 2024-T3 riveted specimens. *Theor Appl FractMech* 2005;43(1):89–99.
- S.A. Fawaz. (2000). Equivalent initial flaw size testing and analysis. 2000.
- Y. Liu and S. Mahadevan. (2008). Probabilistic fatigue life prediction using an equivalent initial flaw size distribution, *International Journal of Fatigue*, Volume 31, Issue 3, March 2009, Pages 476-487, ISSN 0142-1123, DOI: 10.1016/j.ijfatigue.2008.06.005.
- H. Kitagawa, and S. Takahashi. (1976). Applicability of fracture mechanics to very small cracks or cracks in early stage. In: Proceedings of the 2nd international conference on mechanical behavior of materials. USA (OH): ASM International; 1976.
- M.H. El Haddad, T.H. Topper, and K.N. Smith. (1979) Prediction of nonpropagating cracks. *Eng Fract Mech* 1979;11:573–84.
- S.W. Doebling, and F.M. Hemez. (2001) Overview of Uncertainty Assessment for Structural Health Monitoring. In the Proceedings of the 3rd International Workshop on Structural Health Monitoring, September 17-19, 2001, Stanford University, Stanford, California.
- F.M. Hemez, A.N. Roberson, and A.C. Rutherford. (2003). Uncertainty Quantification and Model Validation for Damage Prognosis. In the Proceedings of the 4th International Workshop on Structural Health Monitoring, Stanford University, Stanford, California, September 15-17, 2003
- C.R. Farrar, G. Park, F.M. Hemez, T.B. Tippetts, H. Sohn, J. Wait, D.W. Allen, and B.R. Nadler. (2004). Damage Detection and Prediction for Composite Plates. *J. of The Minerals, Metals and Materials Society*, November 2004.
- C.R. Farrar, and N.A.J. Lieven. (2006). Damage Prognosis : The Future of Structural Health Monitoring. *Phil. Trans. R. Soc. A* (2007) 365, 623–632 doi:10.1098/rsta.2006.1927. Published online 12 December 2006.
- R. Patrick, M.E. Orchard, B. Zhang, M.D. Koelemay, G.J. Kacprzynski, A.A. Ferri, and G.J. Vachtsevanos. (2007). An integrated approach to helicopter planetary gear fault diagnosis and failure prognosis. *Autotestcon, 2007 IEEE*, vol., no., pp.547-552, 17-20 Sept. 2007.
- S. Gupta, and A. Ray. (2007). Real-time fatigue life estimation in mechanical structures. *Meas. Sci. Technol.* 18 (2007) 1947–1957. doi:10.1088/0957-0233/18/7/022.
- S.G. Pierce, K. Worden, and A. Bezazi. (2007). Uncertainty analysis of a neural network used for fatigue lifetime prediction, *Mechanical Systems and Signal Processing*, Volume 22, Issue 6, Special Issue: Mechatronics, August 2008, Pages 1395-1411, ISSN 0888-3270, DOI: 10.1016/j.ymssp.2007.12.004.
- J.M. Papazian, E.L. Anagnostou, S.J. Engel, D. Hoitsma, J. Madsen, R.P. Silberstein, G. Welsh, and J.B. Whiteside. (2008). A structural integrity prognosis system, *Engineering Fracture Mechanics*, Volume 76, Issue 5, Material Damage Prognosis and Life-Cycle Engineering, March 2009, Pages 620-632, ISSN 0013-7944, DOI: 10.1016/j.engfracmech.2008.09.007.
- Y. Liu and S. Mahadevan. (2005). Multiaxial high-cycle fatigue criterion and life prediction for metals. *International Journal of Fatigue*, 2005. 27(7): p. 790-80
- C. Rasmussen. (1996). *Evaluation of Gaussian processes and other methods for non-linear regression*. PhD thesis, University of Toronto, 1996
- T.J. Santner, B.J. Williams, and W.I. Noltz. (2003). *The Design and Analysis of Computer Experiments*. Springer-Verlag, New York, 2003.
- McFarland J. Uncertainty analysis for computer simulations through validation and calibration. Ph D. Dissertation, Vanderbilt University, 2008
- S.A. Richards. (1997). Completed Richardson extrapolation in space and time. *Comm Numer Methods Eng* 1997;13:558–73.

- R. Rebba. (2002) Computational model validation under uncertainty. Master's thesis. Nashville, TN: Vanderbilt University; 2002.
- S.A. Barter, P.K. Sharp, G. Holden, and G. Clark. (2002) Initiation and early growth of fatigue cracks in an aerospace aluminum alloy, *Fatigue Fract. Eng. Mater.* 25 (2002) (2), pp. 111–125.
- A. Makeev, Y. Nikishkov, and E. Armanios. (2007). A concept for quantifying equivalent initial flaw size distribution in fracture mechanics based life prediction models, *Int J Fatigue* (2006), Vol. 29, No. 1, Jan. 2007.
- R. Cross, A. Makeev, and E. Armanios. (2007). Simultaneous uncertainty quantification of fracture mechanics based life prediction model parameters, *Int J Fatigue*, Vol. 29, No. 8, Aug. 2007.
- R. Rebba, S. Mahadevan, and S. Huang. (2004). Validation and error estimation of computational models, *Reliability Engineering & System Safety*, Volume 91, Issues 10-11, The Fourth International Conference on Sensitivity Analysis of Model Output (SAMO 2004) - SAMO 2004, October-November 2006, Pages 1390-1397, ISSN 0951-8320, DOI: 10.1016/j.ress.2005.11.035.
- M. McDonald, K. Zaman, and S. Mahadevan. (2009). Representation and First-Order Approximations for Propagation of Aleatory and Distribution Parameter Uncertainty. In the Proceedings of 50th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, 4 - 7 May 2009, Palm Springs, California.
- A. Haldar, and S. Mahadevan. (2000). *Probability, Reliability and Statistical Methods in Engineering Design*, Wiley, New York, 2000.
- ANSYS. (2007). ANSYS theory reference, release 11.0. ANSYS Inc., 2007.
- Y. Liu, L. Liu, and S. Mahadevan. (2007). Analysis of subsurface crack propagation under rolling contact loading in railroad wheels using FEM. *Engineering Fracture Mechanics*, Vol 74, pgs 2659-2674, 2007.
- G.H. Besterfield, W.K. Liu, A.M. Lawrence, and T. Belytschko. (1991). Fatigue crack growth reliability by probabilistic finite elements, *Computer Methods in Applied Mechanics and Engineering*, Volume 86, Issue 3.
- G.A.F. Seber, and C.J. Wild. 1989. *Nonlinear Regression*. New York: John Wiley and Sons, 1989.
- M. Orchard, G. Kacprzynski, K. Goebel, B. Saha, and G. Vachtsevanos. *Advances in Uncertainty Representation and Management for Particle Filtering Applied to Prognostics*. In the Proceedings of the 1st Prognostics and Health Management (PHM) Conference, Denver, CO. Oct 6-9, 2008.
- P.S. Song, B.C. Sheu, and L. Chang. A modified wheeler model to improve predictions of crack growth following a single overload. *JSME Int J Series A* 2001;44(1):117–22.
- B.K.C. Yuen, and F. Taheri. Proposed modifications to the Wheeler retardation model for multiple overloading fatigue life prediction, *International Journal of Fatigue*, Volume 28, Issue 12, December 2006, Pages 1803-1819, ISSN 0142-1123, DOI: 10.1016/j.ijfatigue.2005.12.007.
- B.C. Sheu, P.S. Song, and S. Hwang. Shaping exponent in wheeler model under a single overload. *Eng Fract Mech* 1995;51(1): 135–43.